USAFA-TR-95-3

A GALERKIN SOLUTION TO GEOMETRICALLY NONLINEAR SHALLOW SHELL EQUATIONS

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APRIL 1995

FINAL REPORT

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6 June 1995

A Galerkin Solution to Geometrically Nonlinear Shallow Shell Equations

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ABSTRACT

A laminated shallow shell approach that includes von Karman geometric nonlinearity and parabolic transverse shear deformation is posed in differential operator form. Trigonometric series are assumed for each of the five shell displacement degrees of freedom for the subsequent nonlinear Galerkin solution resulting in $5n^2$ simultaneous algebraic equations where n is the number of displacement terms assumed in the series. The Galerkin nonlinear solution is computationally intensive. The response of several laminate geometries subjected to transverse loadings are found. Thicker plates and shells generally exhibit more flexible nondimensional displacement response compared to thinner geometries in both linear and nonlinear analyses due to transverse shear deformation. The nondimensional shell response is examined by using the Batdorf-Stein shell parameter for laminated constructions. Quasi-isotropic shallow shells undergo significant transverse shear flexibility in the thicker geometries as given by the nondimensional shell crown deflection. However, the nondimensional crown deflection in the deeper shell response is not much influenced by shell thickness. For flat plates, geometric nonlinearity lessens the influence of transverse shear flexibility when compared to linear solutions due to membrane stretching resistance.

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I. Introduction

The increasingly higher performance demanded of advanced composite materials, especially in weight optimized structures, requires more accurate means of analysis and design. Traditional techniques for thin-walled aerospace structures include classical Kirchhoff plate and Love shell theories. These theories neglect the transverse stress components and therefore are inadequate in many cases due to the sensitivity in laminated materials to transverse shear deformation (Whitney and Pagano, 1970). Consequently, higher order theories have been developed that relax the requirement in the classical approaches where normals to the shell middle plane before deformation remain normal throughout deformation; see for example Lo et al (1977), Kwon and Akin (1987), Whitney and Sun (1974), and the review articles Kapania (1989), Kapania and Reciti (1989), and Noor and Burton (1989, 1990). Generally, Reissner-Mindlin (R-M) (Reissner, 1945, Mindlin, 1951) shear deformation theory requires plane sections remain plane, giving a constant transverse shear through the shell thickness. R-M plate and shell approaches then require shear correction factors that depend on the specific laminate (Whitney, 1973) or the boundary conditions and loading, as described by Pai (1995). More recently, a parabolic distribution of transverse shear through the thickness has been proposed by Levinson (1980), Murthy (1981), and Reddy (1984) for plates and Reddy and Liu (1985) for shells. This approach gives five coupled nonlinear differential equations as in the R-M approaches. Yet, for only a small additional complexity, added accuracy is gained without the ambiguities of shear correction factors. Exact solutions to any of these shell approaches are rare; consequently, approximate techniques are widely used.

Numerical solutions to Reissner-Mindlin theory based on finite element methods are susceptible to shear locking of thin plate/shell meshes unless reduced integration of the stiffness terms (Zienkiewicz et al, 1971, Parisch, 1979) or other numerical fix is performed. (Kui, 1985, Park and Stanley, 1986) The higher order parabolic shear theory including geometric nonlinearity has been solved via finite element techniques for laminated plates by Putcha and Reddy (1986) and laminated shells by Tsai et al (1991). Finite element meshes based on the parabolic shear approach do not shear lock for a wide range of plate/shell geometries.

A related solution approach to the finite element method is the modified Galerkin technique. Giri and Simitses (1980) used this technique to solve geometrically nonlinear plate equations based on classical assumptions. Whitney (1984) gave buckling loads for shallow laminated panels also neglecting transverse shear deformation. Xu and Shen (1986) used the Galerkin method to approximate the classical nonlinear response of laminated plates by assuming quintic B-spline functions. Bowlus et al (1987) used the Galerkin technique to examine the effect of transverse shear flexibility and rotary inertia assuming the R-M approach. Palazotto and Linneman (1991) found buckling loads for laminated shells assuming Sanders equations and parabolic transverse shear distributions through the shell thickness. The influence of transverse shear and radius of curvature is examined. Tighe and Palazotto (1994) assumed Sanders shell equations with nonzero transverse normal strain.

The present study solves laminated shallow shell equations including parabolic transverse shear deformation in the presence of geometric nonlinearity by the modified

Galerkin technique (Sokolnikoff, 1956). The approach is first cast into operator form, resulting in five coupled nonlinear differential equations. Next, a double Fourier series is assumed for each of the displacements and the Galerkin equations are found for simple support boundary conditions. Finally, the response of numerous plate and shell structures, both linear and nonlinear, are presented.

II. Theory

The present study will approximate the response of a shallow cylindrical shell geometry under transverse loading, see Figure 1. Assume the following:

1. the shell is thin such that an approximate state of plane stress exists, and therefore, neglect the transverse normal stress, σ_z, as small in relation to the other stresses.

2. the transverse shear strains, ε_{xz}, ε_{yz}, are distributed parabolically through the thickness of the shell and the transverse shear stresses, σ_{xz}, σ_{yz}, satisfy stress free conditions on the shell's top and bottom surfaces.

3. the shell follows linear elastic laminated material response and therefore is restricted to small strains.

4. the shell geometry is shallow; Donnell shallow shell strain-displacement relations are assumed.

5. the shell geometry and the magnitude of the loading may result in large deflections; therefore, assume geometric nonlinearity in the von Karman sense.

As mentioned in the introduction, numerous plate and shell studies have shown that due to the relative weakness in shear of current high stiffness-high strength laminated materials, transverse shear deformation can be a significant influence even for relatively thin geometries as compared to similar isotropic shells. Consequently, the present approach will neglect the transverse normal stress, σ_z, as small compared to the shell's inplane stresses, σ_x, σ_y, σ_{xy}, as is typical in plane stress assumptions. However, the approach will approximately account for transverse shear deformation by retaining the transverse shear stresses, σ_{xz} and σ_{yz}. This approach is consistent with shell stress order of magnitude studies by Koiter (1967) and John (1965) for isotropic shells. Additionally, exact solutions of laminated flat plates (Pagano, 1970) and cylindrical shells (Ren, 1987) in cylindrical bending show this to be a reasonable approximation. Furthermore, since σ_z generally is of order h/R times the bending stress and σ_{xz} and σ_{yz} are of order h/r or h/s times the bending stress, σ_z is negligible compared to σ_{xz} and σ_{yz} when $r,s \ll R$. (Reddy, 1985, Palazotto and Dennis, 1992)

Classical Donnell shallow shell assumptions restrict $R/h{>}500$, (Whitney, 1984). However, Palazotto and Linneman (1991) show that for shell vibration and buckling, the added transverse shear degrees of freedom allow accuracy to approximately $R/h{>}50$.

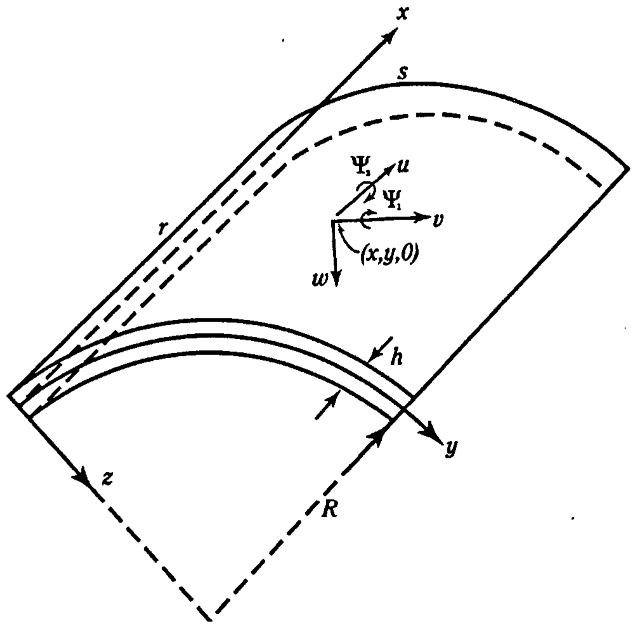

FIGURE 1. Shell geometry of radius R, thickness h, planform r x s.

KINEMATICS AND STRAIN

Assuming truncated power series in the transverse coordinate, z, for the shell displacements and satisfying stress-free conditions on the shell's top and bottom surfaces results in the cubic kinematics of Eqn (1). Eqn (1) gives parabolic approximations for the transverse shear strains through the shell thickness. Kinematics of Eqn (1) and similar forms have been studied by several investigators (Levinson 1980, Murthy 1981, Reddy 1984, Bhimaraddi 1984, Reddy and Liu 1985, Soldatos 1987, Dennis and Palazotto 1990) applying them to both laminated flat plate and shell constructions in both linear and nonlinear, buckling, and vibrational analyses.

$$
\begin{aligned}
u_1(x,y,z) &= u + z\Psi_1 + z^3 k_h(\Psi_1 + w,_x) \\
u_2(x,y,z) &= v + z\Psi_2 + z^3 k_h(\Psi_2 + w,_y) \\
u_3(x,y) &= w \\
k_h &\equiv \frac{-4}{3h^2}
\end{aligned}
\tag{1}
$$

where, u_1, u_2, u_3 are the shell displacements in the x, y, z, coordinate directions respectively; u, v, w are shell midplane displacements (i.e., at z=0); and Ψ_1, Ψ_2 are

rotations of the normals to the shell midplane about the y, x axis respectively, and commas denote partial differentiation with respect to the coordinate shown.

Donnell strain-displacement relations including a simple geometric nonlinearity in the von Karman sense are given below in Eqn (2) (Brush and Almroth, 1975). The nonlinearity assumes the strains and rotations squared are small compared to one and allow deflections of magnitude the same order as the shell thickness (Chia, 1988). The typical stress and strain shorthand notation defined in Eqn (2) is adopted.

$$\begin{aligned}
\varepsilon_x &\equiv \varepsilon_1 = u_1,_x + \tfrac{1}{2} w,_x^2 \\
\varepsilon_y &\equiv \varepsilon_2 = u_2,_y - \frac{w}{R} + \tfrac{1}{2} w,_y^2 \\
\varepsilon_{xy} &\equiv \varepsilon_6 = u_1,_y + u_2,_x + w,_x\, w,_y \\
\varepsilon_{yz} &\equiv \varepsilon_4 = u_2,_z + u_3,_y \\
\varepsilon_{xz} &\equiv \varepsilon_5 = u_3,_x + u_1,_z
\end{aligned} \tag{2}$$

The Donnell-von Karman strain-displacement relations of Eqn (2) together with the shell kinematics of Eqn (1) give the shell strain-displacement relations in Eqn (3).

$$\begin{aligned}
\varepsilon_1 &= \overset{\circ}{\varepsilon}_1 + z\kappa_{11} + z^3\kappa_{13} \\
\varepsilon_2 &= \overset{\circ}{\varepsilon}_2 + z\kappa_{21} + z^3\kappa_{23} \\
\varepsilon_6 &= \overset{\circ}{\varepsilon}_6 + z\kappa_{61} + z^3\kappa_{63} \\
\varepsilon_4 &= \overset{\circ}{\varepsilon}_4 + z^2\kappa_{42} \\
\varepsilon_5 &= \overset{\circ}{\varepsilon}_5 + z^2\kappa_{52}
\end{aligned} \tag{3}$$

where the midplane strains are given by,

$$\begin{aligned}
\overset{\circ}{\varepsilon}_1 &= u,_x + \tfrac{1}{2} w,_x^2 \\
\overset{\circ}{\varepsilon}_2 &= v,_y - \frac{w}{R} + \tfrac{1}{2} w,_y^2 \\
\overset{\circ}{\varepsilon}_6 &= u,_y + v,_x + w,_x\, w,_y \\
\overset{\circ}{\varepsilon}_4 &= \Psi_2 + w,_y \\
\overset{\circ}{\varepsilon}_5 &= \Psi_1 + w,_x
\end{aligned}$$

and the curvature strains are given by,

$$\kappa_{11}=\Psi_{1},_{x} \qquad \kappa_{13}=k_h(\Psi_{1},_{x}+w,_{xx}) \qquad \kappa_{42}=3k_h(\Psi_2+w,_{y})$$
$$\kappa_{21}=\Psi_{2},_{y} \qquad \kappa_{23}=k_h(\Psi_{2},_{y}+w,_{yy}) \qquad \kappa_{52}=3k_h(\Psi_1+w,_{x})$$
$$\kappa_{61}=\Psi_{1},_{y}+\Psi_{2},_{x} \qquad \kappa_{63}=k_h(\Psi_{1},_{y}+\Psi_{2},_{x}+2w,_{xy})$$

CONSTITUTIVE RELATIONS

Consider shells constructed of layers of unidirectional fibers imbedded in a matrix, i.e., transversely isotropic material. (Jones, 1975) The three dimensional stress-strain relationships for a single lamina of transversely isotropic material are reduced to two dimensions using the plane stress assumption. By neglecting the transverse normal stress (σ_z=σ_3=0), the transverse normal strain (ε_3) becomes dependent on the in-plane strains (ε_1,ε_2). Eliminating ε_3 then gives stress-strain relationships in material coordinates where the fibers of the lamina are aligned with the x coordinate, within an individual ply of a laminated shell as shown in Eqn (4). Shell structures are layered with the material governed by Eqn (4) where the fibers may be oriented arbitrarily within the plane of the shell, i.e., the x-y plane. Transforming Eqn (4) from material coordinates to shell coordinates then gives Eqn (5) for the nth ply of the laminate.

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\\ \sigma_4\\ \sigma_5\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\ Q_{12} & Q_{22} & 0 & 0 & 0\\ 0 & 0 & Q_{66} & 0 & 0\\ 0 & 0 & 0 & Q_{44} & 0\\ 0 & 0 & 0 & 0 & Q_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\\ \varepsilon_4\\ \varepsilon_5\end{Bmatrix} \tag{4}$$

where the stress notation is defined analogously to the strains, see Eqn (2) and the Q_{ij} are elements of a symmetric array of the reduced stiffnesses and can be written in terms of the engineering constants as given below,

$$Q_{11}=\frac{E_1}{\Delta}, Q_{22}=\frac{E_2}{\Delta}, Q_{12}=\upsilon_{12}\frac{E_2}{\Delta}, Q_{66}=G_{12}$$
$$Q_{44}=G_{23}, Q_{55}=G_{13}, \Delta=1-\upsilon_{21}\upsilon_{12}$$

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}^n = \begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ & \overline{Q}_{22} & \overline{Q}_{26}\\ & & \overline{Q}_{66}\end{bmatrix}^n \begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix}, \quad \begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix}^n = \begin{bmatrix}\overline{Q}_{44} & \overline{Q}_{45}\\ & \overline{Q}_{55}\end{bmatrix}^n \begin{Bmatrix}\varepsilon_4\\ \varepsilon_5\end{Bmatrix} \tag{5}$$

where the $\overline{Q}_{ij}$ are elements of a symmetric array of the transformed reduced stiffnesses and are functions of the reduced stiffnesses and angle of orientation of the fibers of the nth ply. (Jones,1975)

A two dimensional shell approach will "smear" or average the constitutive relations of each ply by integrating Eqn (5) through the shell thickness. First substitute the strains of Eqn (3) into Eqn (5), see Eqns (6). Using averaged load quantities defined in Eqns (7), the load-strain relations of Eqns (8) result where in Eqns (8), repeated indices imply summation, i,j = 1,2,6; m,n = 4,5; and the elasticity arrays are defined in Eqn (9).

$$\begin{gathered}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}^n = \begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ & \overline{Q}_{22} & \overline{Q}_{26}\\ & & \overline{Q}_{66}\end{bmatrix}^n \left[\begin{Bmatrix}\varepsilon_1^\circ\\ \varepsilon_2^\circ\\ \varepsilon_6^\circ\end{Bmatrix} + z\begin{Bmatrix}\kappa_{11}\\ \kappa_{21}\\ \kappa_{61}\end{Bmatrix} + z^3\begin{Bmatrix}\kappa_{13}\\ \kappa_{23}\\ \kappa_{63}\end{Bmatrix}\right]\\ \begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix}^n = \begin{bmatrix}\overline{Q}_{44} & \overline{Q}_{45}\\ & \overline{Q}_{55}\end{bmatrix}^n \left[\begin{Bmatrix}\varepsilon_4^\circ\\ \varepsilon_5^\circ\end{Bmatrix} + z^2\begin{Bmatrix}\kappa_{42}\\ \kappa_{52}\end{Bmatrix}\right]\end{gathered} \tag{6}$$

$$\begin{gathered}\begin{Bmatrix}N_1\\ N_2\\ N_6\end{Bmatrix}, \begin{Bmatrix}M_1\\ M_2\\ M_6\end{Bmatrix}, \begin{Bmatrix}P_1\\ P_2\\ P_6\end{Bmatrix} \equiv \int_{-h/2}^{h/2} \begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}(1, z, z^3)dz,\\ \begin{Bmatrix}Q_4\\ Q_5\end{Bmatrix}, \begin{Bmatrix}R_4\\ R_5\end{Bmatrix} \equiv \int_{-h/2}^{h/2} \begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix}(1, z^2)dz\end{gathered} \tag{7}$$

$$\begin{aligned}N_i &= A_{ij}\varepsilon_j^\circ + B_{ij}\kappa_{j1} + E_{ij}\kappa_{j3} & Q_m &= A_{mn}\varepsilon_n^\circ + D_{mn}\kappa_{n2}\\ M_i &= B_{ij}\varepsilon_j^\circ + D_{ij}\kappa_{j1} + F_{ij}\kappa_{j3} & R_m &= D_{mn}\varepsilon_n^\circ + F_{mn}\kappa_{n2}\\ P_i &= E_{ij}\varepsilon_j^\circ + F_{ij}\kappa_{j1} + H_{ij}\kappa_{j3} & &\end{aligned} \tag{8}$$

$$\left(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij}\right)\equiv\int_{-h/2}^{h/2}\overline{Q}_{ij}\left(1,z,z^2,z^3,z^4,z^6\right)dz$$
$$\left(A_{mn},D_{mn},F_{mn}\right)\equiv\int_{-h/2}^{h/2}\overline{Q}_{mn}\left(1,z^2,z^4\right)dz \tag{9}$$

VIRTUAL WORK AND EQUILIBRIUM

The principle of virtual displacements for the shell loaded transversely by traction, q, is stated as,

$$\int_h\left\{\iint_{\Omega_n}\left(\sigma_1\cdot\delta\varepsilon_1^n+\sigma_2\cdot\delta\varepsilon_2^n+\sigma_6\cdot\delta\varepsilon_6^n+\sigma_4\cdot\delta\varepsilon_4^n+\sigma_5\cdot\delta\varepsilon_5^n\right)d\Omega_n\right\}dz-\int_\Omega q\delta w d\Omega=0 \tag{10}$$

where Ω_n is the shell's x-y plane within the nth ply.

Substituting the quantities defined in Eqns (3) and (7), the virtual work expression can be written in two dimensions as is shown in Eqn (11).

$$\begin{aligned}\iint_\Omega\ &(N_1\cdot\delta\varepsilon_1^\circ+M_1\cdot\delta\kappa_{11}+P_1\cdot\delta\kappa_{13}+N_2\cdot\delta\varepsilon_2^\circ+M_2\cdot\delta\kappa_{21}+P_2\cdot\delta\kappa_{23}+\\ &N_6\cdot\delta\varepsilon_6^\circ+M_6\cdot\delta\kappa_{61}+P_6\cdot\delta\kappa_{63}+Q_4\cdot\delta\varepsilon_4^\circ+R_4\cdot\delta\kappa_{42}+\\ &Q_5\cdot\delta\varepsilon_5^\circ+R_5\cdot\delta\kappa_{52}-q\delta w)d\Omega=0\end{aligned} \tag{11}$$

The shell equilibrium equations are found from Eqn (11) by substituting the strain-displacement equations of (3) and then integrating by parts. Since the variation of each of the five displacement functions is arbitrary and independent, their coefficients can each individually be set to zero. With this in mind, terms were grouped for each virtual displacement and the resulting five equations are given in Eqn (12). The five equilibrium equations are a subset of Eqns (12) where equilibrium within the shell domain is described by each double integral term vanishing. The remaining terms in each equation are the boundary conditions for the shell where along the indicated shell edge, either displacement is prescribed or the force term is zero. The given form of Eqns (12) will be useful for the subsequent Galerkin solution. The underlined terms of Eqn (12c) are known as "buoyancy" terms (Chia, 1980) and vanish due to the inplane equilibrium of Eqns (12a,b).

Excluding the nonlinear terms of Eqn (12), i.e., those where a force quantity multiplies a displacement, gives equilibrium equations similar to those of other investigators. (Reddy and Liu, 1985; Linneman and Palazotto, 1991)

$$\iint_{\Omega} \left(N_{1,x} + N_{6,y}\right)\delta u \, d\Omega - \int_0^r N_6 \delta u\Big|_0^s dx - \int_0^s N_1 \delta u\Big|_0^r dy = 0 \tag{12a}$$

$$\iint_{\Omega} \left(N_{2,y} + N_{6,x}\right)\delta v \, d\Omega - \int_0^s N_6 \delta v\Big|_0^r dy - \int_0^r N_2 \delta v\Big|_0^s dx = 0 \tag{12b}$$

$$\begin{aligned}
&-\iint_{\Omega} (N_1 w_{,xx} + \underline{N_{1,x} w_{,x}} + N_2/R + N_2 w_{,yy} + \underline{N_{2,y} w_{,y}} + 2N_6 w_{,xy} + \underline{N_{6,x} w_{,y}} + \underline{N_{6,y} w_{,x}} \\
&\quad - P_{1,xx} k_h - P_{2,yy} k_h - 2P_{6,xy} k_h + Q_{4,y} + Q_{5,x} + 3k_h R_{4,y} + 3k_h R_{5,x})\delta w \, d\Omega + \\
&\int_0^s \left(N_1 w_{,x} + N_6 w_{,y} - P_{1,x} k_h - 2P_{6,y} k_h + Q_5 + 3k_h R_5\right)\delta w\Big|_0^r dy + \\
&\int_0^r \left(N_2 w_{,y} + N_6 w_{,x} - P_{2,y} k_h - 2P_{6,x} k_h + Q_4 + 3k_h R_4\right)\delta w\Big|_0^s dx + \\
&2k_h P_6 \delta w\Big|_0^r\Big|_0^s + \int_0^s k_h P_1 \delta w_{,x}\Big|_0^r dy + \int_0^r k_h P_2 \delta w_{,y}\Big|_0^s dx = \iint_{\Omega} q \delta w \, d\Omega
\end{aligned} \tag{12c}$$

$$\begin{aligned}
&\iint_{\Omega} \left(M_{1,x} + M_{6,y} + k_h P_{1,x} + k_h P_{6,y} - Q_5 - 3k_h R_5\right)\delta \Psi_1 \, d\Omega \\
&-\int_0^s \left(M_1 + k_h P_1\right)\delta \Psi_1\Big|_0^r dy - \int_0^r \left(M_6 + k_h P_6\right)\delta \Psi_1\Big|_0^s dx = 0
\end{aligned} \tag{12d}$$

$$\begin{aligned}
&\iint_{\Omega} \left(M_{2,y} + M_{6,x} + k_h P_{2,y} + k_h P_{6,x} - Q_4 - 3k_h R_4\right)\delta \Psi_2 \, d\Omega \\
&-\int_0^s \left(M_6 + k_h P_6\right)\delta \Psi_2\Big|_0^r dy - \int_0^r \left(M_2 + k_h P_2\right)\delta \Psi_2\Big|_0^s dx = 0
\end{aligned} \tag{12e}$$

III. Galerkin Solution

The Galerkin technique is one type of the general methods of weighted residuals used to find approximate solutions to differential equations (Sokolnikoff, 1956). In this section, the Galerkin technique is described first generally and then the technique is applied to the shell equations of (12).

Assume a differential equation of the form shown in Eqn (13)

$$Lu - f = 0 \tag{13}$$

where L is a differential operator, u is the field variable (displacement for the present study), and f is the loading. Next, approximate u by $\bar{u}$ as in Eqn (14)

$$\bar{u} = c_j \phi_j,\ j = 1,2,3 \ldots N \tag{14}$$

where there is an implied summation on j, the c_j are unknown constants, and the ϕ_j are coordinate functions that satisfy the boundary conditions. Substituting the approximate displacement of Eqn (14) into (13) then gives,

$$L\bar{u} - f = \xi \tag{15}$$

In Eqn (15) the differential equation does not equal zero; but instead equals some error due to the approximation on the field variable. In the weighted residual technique, the constants c_j of the approximation of Eqn (15), are found such that the error in Eqn (15) is forced to zero in an average sense by integrating Eqn (15) multiplied by weighting functions over the domain as in Eqn (16).

$$\int_D \left(L\bar{u} - f\right)\psi_i \, dD = 0,\ i=1,2,\ldots N \tag{16}$$

The case where the weighting functions, ψ_i, are identical to the coordinate functions of Eqn (14) is the Galerkin method. Eqn (16) represents N simultaneous equations in the N unknowns, c_j. Therefore, a single differential equation is approximated to desired accuracy by N algebraic equations.

The present shallow shell approach can be posed in the operator form of Eqn (13) as given below in Eqns (17)-(19). In the equations, the force-displacement relations of Eqn (8) with Eqn (3) are substituted into the equilibrium equations of (12) giving five

coupled partial differential equations in displacement. A significant simplification results for symmetrically stacked laminates where the elasticity arrays of Eqn (9) associated with odd powers of z are zero. The symbolic math package, *Mathematica* (Wolfram Research Inc. 1993), performed the extensive algebra.

$$(L+H)d=f \tag{17}$$

where L and H are the linear and nonlinear operators, respectively, and are defined as,

$$L \equiv \begin{bmatrix} L_{11} & L_{12} & L_{13} & 0 & 0 \\ L_{12} & L_{22} & L_{23} & 0 & 0 \\ L_{13} & L_{23} & L_{33} & L_{34} & L_{35} \\ 0 & 0 & L_{34} & L_{44} & L_{45} \\ 0 & 0 & L_{35} & L_{45} & L_{55} \end{bmatrix} \tag{18}$$

$$H \equiv \begin{bmatrix} 0 & 0 & H_{13} & 0 & 0 \\ 0 & 0 & H_{23} & 0 & 0 \\ 0 & 0 & -H_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

and the elements of the operators are defined below,

$$L_{11}(\;)=A_{11}(\;)_{,xx}+2A_{16}(\;)_{,xy}+A_{66}(\;)_{,yy}$$
$$L_{12}(\;)=A_{16}(\;)_{,xx}+[A_{12}+A_{66}](\;)_{,xy}+A_{26}(\;)_{,yy}$$
$$L_{13}(\;)=-\frac{A_{12}}{R}(\;)_{,x}-\frac{A_{26}}{R}(\;)_{,y}$$

$$L_{22}(\;)=A_{66}(\;)_{,xx}+2A_{26}(\;)_{,xy}+A_{22}(\;)_{,yy}$$
$$L_{23}(\;)=-\frac{A_{26}}{R}(\;)_{,x}-\frac{A_{22}}{R}(\;)_{,y}$$

$$L_{33}(\,)=\frac{A_{22}}{R^2}(\,)-\left[2A_{45}+12D_{45}k_h+18F_{45}k_h{}^2\right](\,)_{,xy}-\left[A_{55}+6D_{55}k_h+9F_{55}k_h{}^2\right](\,)_{,xx}$$
$$-\left[A_{44}+6D_{44}k_h+9F_{44}k_h{}^2\right](\,)_{,yy}+H_{11}k_h{}^2(\,)_{,xxxx}+2H_{12}k_h{}^2(\,)_{,xxyy}+H_{22}k_h{}^2(\,)_{,yyyy}$$
$$+4H_{16}k_h{}^2(\,)_{,xxxy}+4H_{26}k_h{}^2(\,)_{,xyyy}+4H_{66}k_h{}^2(\,)_{,xxyy}$$

$$L_{34}(\,)=-\left[A_{55}+6D_{55}k_h+9F_{55}k_h{}^2\right](\,)_{,x}-\left[A_{45}+6D_{45}k_h+9F_{45}k_h{}^2\right](\,)_{,y}$$
$$+\left[F_{11}k_h+H_{11}k_h{}^2\right](\,)_{,xxx}+\left[F_{26}k_h+H_{26}k_h{}^2\right](\,)_{,yyy}+\left[3F_{16}k_h+3H_{16}k_h{}^2\right](\,)_{,xxy}$$
$$+\left[F_{12}k_h+2F_{66}k_h+H_{12}k_h{}^2+2H_{66}k_h{}^2\right](\,)_{,xyy}$$

$$L_{35}(\,)=-\left[A_{45}+6D_{45}k_h+9F_{45}k_h{}^2\right](\,)_{,x}-\left[A_{44}+6D_{44}k_h+9F_{44}k_h{}^2\right](\,)_{,y}$$
$$+\left[F_{16}k_h+H_{16}k_h{}^2\right](\,)_{,xxx}+\left[F_{22}k_h+H_{22}k_h{}^2\right](\,)_{,yyy}$$
$$+\left[F_{12}k_h+2F_{66}k_h+H_{12}k_h{}^2+2H_{66}k_h{}^2\right](\,)_{,xxy}+\left[3F_{26}k_h+3H_{26}k_h{}^2\right](\,)_{,xyy}$$

$$L_{44}(\,)=-\left[A_{55}+6D_{55}k_h+9F_{55}k_h{}^2\right](\,)+\left[D_{11}+2F_{11}k_h+H_{11}k_h{}^2\right](\,)_{,xx}$$
$$+\left[2D_{16}+4F_{16}k_h+2H_{16}k_h{}^2\right](\,)_{,xy}+\left[D_{66}+2F_{66}k_h+H_{66}k_h{}^2\right](\,)_{,yy}$$

$$L_{45}(\,)=-\left[A_{45}+6D_{45}k_h+9F_{45}k_h{}^2\right](\,)+\left[D_{16}+2F_{16}k_h+H_{16}k_h{}^2\right](\,)_{,xx}$$
$$+\left[D_{12}+D_{66}+2F_{12}k_h+2F_{66}k_h+H_{12}k_h{}^2+H_{66}k_h{}^2\right](\,)_{,xy}$$
$$+\left[D_{26}+2F_{26}k_h+H_{26}k_h{}^2\right](\,)_{,yy}$$

$$L_{55}(\,)=-\left[A_{44}+6D_{44}k_h+9F_{44}k_h{}^2\right](\,)+\left[D_{66}+2F_{66}k_h+H_{66}k_h{}^2\right](\,)_{,xx}$$
$$+\left[2D_{26}+4F_{26}k_h+2H_{26}k_h{}^2\right](\,)_{,xy}+\left[D_{22}+2F_{22}k_h+H_{22}k_h{}^2\right](\,)_{,yy}$$

$$H_{13}(\,)=w_{,x}\,L_{11}(\,)+w_{,y}\,L_{12}(\,)$$

$$H_{23}(\,)=w_{,x}\,L_{12}(\,)+w_{,y}\,L_{22}(\,)$$

$$H_{33}(\,)=\left(u_{,x}+\frac{1}{2}w_{,x}^2\right)L_1(\,)+\left(v_{,y}-\frac{w}{R}+\frac{1}{2}w_{,y}^2\right)L_2(\,)+$$
$$\left(u_{,y}+v_{,x}+w_{,x}\,w_{,y}\right)L_3(\,)-\frac{1}{2}w_{,x}\,L_{13}(\,)-\frac{1}{2}w_{,y}\,L_{23}(\,)$$

$$L_1(\,)=A_{11}(\,)_{,xx}+2A_{16}(\,)_{,xy}+A_{12}(\,)_{,yy}$$

$$L_2(\ ) = A_{12}(\ )_{,xx} + 2A_{26}(\ )_{,xy} + A_{22}(\ )_{,yy}$$

$$L_3(\ ) = A_{16}(\ )_{,xx} + 2A_{66}(\ )_{,xy} + A_{26}(\ )_{,yy}$$

The displacement and load vector are given by,

$$d = \begin{Bmatrix} u \\ v \\ w \\ \Psi_1 \\ \Psi_2 \end{Bmatrix}, \quad f = \begin{Bmatrix} 0 \\ 0 \\ q \\ 0 \\ 0 \end{Bmatrix} \tag{20}$$

SIMPLY-SUPPORTED BOUNDARY CONDITIONS

Consider the simply-supported boundary conditions given below in Eqn (21). These give stress-free membrane conditions normal to the shell plane edges; yet, stress may develop tangentially along each edge due to the restrained tangential displacement.

$$\begin{gathered} \text{along edges } x=0,\ x=r: \\ v = w = \Psi_2 = 0 \\ \\ \text{along edges } y=0,\ y=s: \\ u = w = \Psi_1 = 0 \end{gathered} \tag{21}$$

In the Galerkin approach, the displacements of the shell are approximated in the form of Eqn (14) below in Eqn (22). The coordinate functions must satisfy the geometric boundary conditions of Eqn (21) and presently, a double Fourier series is assumed for each of the five.

$$\begin{aligned} u &= \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\phi_{mn} \\ v &= \sum_{m=1}^{M}\sum_{n=1}^{N} b_{mn}\chi_{mn} \\ w &= \sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}\alpha_{mn} \\ \Psi_1 &= \sum_{m=1}^{M}\sum_{n=1}^{N} d_{mn}\gamma_{mn} \\ \Psi_2 &= \sum_{m=1}^{M}\sum_{n=1}^{N} e_{mn}\eta_{mn} \end{aligned} \tag{22}$$

where the coordinate functions are,

$$\phi_{mn} = \cos\frac{m\pi x}{r}\sin\frac{n\pi y}{s}$$

$$\chi_{mn} = \sin\frac{m\pi x}{r}\cos\frac{n\pi y}{s}$$

$$\alpha_{mn} = \sin\frac{m\pi x}{r}\sin\frac{n\pi y}{s}$$

$$\gamma_{mn} = \cos\frac{m\pi x}{r}\sin\frac{n\pi y}{s}$$

$$\eta_{mn} = \sin\frac{m\pi x}{r}\cos\frac{n\pi y}{s}$$

Substituting Eqn (22) into Eqn (17) plus reintroducing the nonzero boundary condition terms of Eqn (12), gives the Galerkin equations of the form of Eqn (16) in Eqns (23). Since admissible functions are assumed for the displacements, i.e., those that only satisfy the geometric boundary conditions, the single integral terms are the remaining nonzero boundary conditions and must be retained as is shown.

$$\begin{aligned}&\int_0^r\int_0^s [L_{11}(u)+L_{12}(v)+L_{13}(w)+H_{13}(w)]\phi_{pq}\,dxdy + \\ &\qquad \int_0^s N_1(0,y)\phi_{pq}(0,y)\,dy - \int_0^s N_1(r,y)\phi_{pq}(r,y)\,dy = 0\end{aligned} \tag{23a}$$

$$\begin{aligned}&\int_0^r\int_0^s [L_{12}(u)+L_{22}(v)+L_{23}(w)+H_{23}(w)]\chi_{pq}\,dxdy + \\ &\qquad \int_0^r N_2(x,0)\chi_{pq}(x,0)\,dx - \int_0^r N_2(x,s)\chi_{pq}(x,s)\,dx = 0\end{aligned} \tag{23b}$$

$$\begin{aligned}&\int_0^r\int_0^s [L_{13}(u)+L_{23}(v)+L_{33}(w)+L_{34}(\Psi_1)+L_{35}(\Psi_2)-H_{33}(w)]\alpha_{pq}\,dxdy + \\ &\qquad k_h\int_0^s P_1(r,y)\alpha_{pq},_x(r,y)\,dy - k_h\int_0^s P_1(0,y)\alpha_{pq},_x(0,y)\,dy + \\ &\qquad k_h\int_0^r P_2(x,s)\alpha_{pq},_y(x,s)\,dx - k_h\int_0^r P_2(x,0)\alpha_{pq},_y(x,0)\,dx = \int_0^r\int_0^s q\alpha_{pq}\,dxdy\end{aligned} \tag{23c}$$

$$\int_0^r\int_0^s [L_{34}(w)+L_{44}(\Psi_1)+L_{45}(\Psi_2)]\gamma_{pq}dxdy +$$

$$\int_0^s M_1(0,y)\gamma_{pq}(0,y)\,dy - \int_0^s M_1(r,y)\gamma_{pq}(r,y)\,dy + \tag{23d}$$

$$k_h\int_0^s P_1(0,y)\gamma_{pq}(0,y)\,dy - k_h\int_0^s P_1(r,y)\gamma_{pq}(r,y)\,dy = 0$$

$$\int_0^r\int_0^s [L_{35}(w)+L_{45}(\Psi_1)+L_{55}(\Psi_2)]\eta_{pq}dxdy +$$

$$\int_0^r M_2(x,0)\eta_{pq}(x,0)\,dx - \int_0^r M_2(x,s)\eta_{pq}(x,s)\,dx \tag{23e}$$

$$k_h\int_0^r P_2(x,0)\eta_{pq}(x,0)\,dx - k_h\int_0^r P_2(x,s)\eta_{pq}(x,s)\,dx = 0$$

Eqns (22) are next substituted into Eqns (23) giving a system of $5MN$ simultaneous nonlinear algebraic equations in the unknown constants a_{mn}, b_{mn}, c_{mn}, d_{mn}, e_{mn}. In getting to the simultaneous equations, many trigonometric integrals must be evaluated. The general forms are shown in Appendix A. The final form of the Galerkin equations given as functions of displacement is found in Appendix B. Appendix C is a listing of the Fortran coding. In addition, sample input and output listings are included.

LOADING

Various transverse loadings can be considered by first expanding the traction q of the right hand side of Eqn (23c) into a Fourier series as shown in Eqn (24).

$$q = q(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} p_{mn}\sin\frac{m\pi x}{r}\sin\frac{n\pi y}{s} \tag{24}$$

Following Timoshenko and Woinowsky-Kreiger (1959), any coefficient of the series can be found from:

$$p_{ij} = \frac{4}{rs}\int_0^r\int_0^s q(x,y)\sin\frac{i\pi x}{r}\sin\frac{j\pi y}{s}dxdy \tag{25}$$

The following transverse loadings are then derived from Eqns (24) and (25),

Uniform loading of magnitude q_o:

$$q = \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{16q_{\circ}}{\pi^2 mn} \sin\frac{m\pi x}{r} \sin\frac{n\pi y}{s} \tag{26}$$

Sinusoidal loading of magnitude $q_{\circ}$:

$$q = q_{\circ} \sin\frac{\pi x}{r} \sin\frac{\pi y}{s} \tag{27}$$

Point load, $p_{\circ}$ applied at shell coordinates $(x_{\circ},y_{\circ})$:

$$q = \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{4p_{\circ}}{rs} \sin\frac{m\pi x_{\circ}}{r} \sin\frac{n\pi y_{\circ}}{s} \sin\frac{m\pi x}{r} \sin\frac{n\pi y}{s} \tag{28}$$

IV. Linear Plate and Shell Solutions

As a first step in verifying the solution procedure, several linear cases are presented and compared to published results. Linear solutions assume proportional load and deflection and therefore the terms of the operator $\boldsymbol{H}$ in Eqn (17) are not included. For many of the cases examined, biaxial symmetry is present and consequently only the odd terms of the assumed displacement functions of Eqn (22), i.e., m,n = 1,3,5...etc., contribute to the solutions. Quasi-isotropic plates and shells, such as the $[0/\pm45/90]_s$ and $[-60/0/60]_s$ laminates examined in this section, have nonzero D_{16} and D_{26} terms for example that destroy the symmetry and in that case, both even and odd displacement functions are required. Unique results are given for a $[-60/0/60]_s$ shell laminate using the recently proposed Batdorf-Stein parameter.

ISOTROPIC PLATE

Consider a thin isotropic plate subjected to two different transverse loadings, uniform pressure q, and a single point load p_o applied at the center of the plate, see Eqns (26) and (28). For a thin geometry, i.e. for this case, the thickness ratio $S=s/h=100$, transverse shear deformation is negligible and comparison to the classical thin plate response should be close. Table 1 gives the center plate transverse displacement solution for an increasing number of terms M,N in the assumed solution of Eqn (22). As mentioned, for uniform and center plate point loads and isotropic material behavior, only the symmetric coefficient functions, i.e., odd m,n in Eqn (22) contribute to the solution. Consequently, m is 1,3,5,...M and n is 1,3,5,...N. Both of the uniform load cases converge to within 0.1% of the classical result taken from Pilkey and Chang (1978) after only a few terms in the series. The point load case requires several additional terms.

CROSS-PLY LAMINATED PLATE

Results are shown for square cross-ply laminates, [0/90/0] and $[0/90]_s$. The [0/90/0] laminate is subjected to a uniform load of Eqn (26) and the nondimensionalized center plate deflection of Eqn (29a) requires several terms for convergence, see Table 2. The increasing flexibility due to transverse shear deformation is evident in Table 2 for the smaller thickness ratios, S. For M,N=21 and odd m,n, the present Galerkin solution exactly matches the results due to Reddy (1984).

The $[0/90]_s$ laminate is subjected to the sinusoidal pressure loading of Eqn (27) and one term (M,N=1), due to this special loading case, gives the exact solution. Table 3 shows the nondimensionalized deflection and stress of Eqns (29). All results of Table 3 are identical to those given by Reddy (1984). Transverse shear deformation is again evident in the thick plates.

TABLE 1. Center plate deflection for an isotropic thin plate. Deflection is nondimensionalized by $100\,D/qa^4$ for uniform loads and by $100\,D/Pa^2$ for the point load case, where $D=Eh^3/12(1-\upsilon^2)$, υ=0.3; Square: $r=s=a$; rectangular: $r=2s=2a$.

M, N	square uniform load	rectangular uniform load	square point load
1	0.41607	1.06514	1.02720
5	0.40637	1.01387	1.14332
9	0.40624	1.01297	1.15485
13	0.40624	1.01288	1.15815
29		1.01287	1.15897
classical (M,N =99 in Pilkey, 1978)	0.40624	1.01287	1.16002

$$\overline{w} = \frac{100E_2h^3}{qr^4} w\left(\frac{r}{2},\frac{s}{2}\right) \tag{29a}$$

$$\overline{\sigma}_1 = \left|\frac{h^2}{qr^2}\sigma_1\left(\frac{r}{2},\frac{s}{2},-\frac{h}{2}\right)\right| \tag{29b}$$

$$\overline{\sigma}_2 = \left|\frac{h^2}{qr^2}\sigma_2\left(\frac{r}{2},\frac{s}{2},-\frac{h}{4}\right)\right| \tag{29c}$$

$$\overline{\sigma}_4 = \left|\frac{h}{qr}\sigma_4\left(\frac{r}{2},0,0\right)\right| \tag{29d}$$

$$\overline{\sigma}_5 = \left|\frac{h}{qr}\sigma_5\left(0,\frac{s}{2},0\right)\right| \tag{29e}$$

where in Eqns (29), from Figure 1, r and s represent the shell planform, h is the shell thickness, and q is defined in Eqns (24)-(28); the quantities are evaluated at the plate coordinates given in parentheses.

TABLE 2. Nondimensionalized center plate deflection for [0/90/0] laminate subjected to uniform load; E_1/E_2=25, G_{12}/E_2=G_{13}/E_2=0.5, G_{23}/E_2=0.2, υ_{12}=0.25.

$S=s/h$	M,N 1	5	13	21	Reddy (1984)
100	0.7039	0.6709	0.6705	0.6705	0.6705
50	0.7182	0.6843	0.6838	0.6838	0.6838
20	0.8173	0.7769	0.7760	0.7760	0.7760
10	1.1551	1.0921	1.0901	1.0900	1.0900
4	3.1156	2.9167	2.9094	2.9091	2.9091
2	8.3141	7.7823	7.7665	7.7661	7.7661

TABLE 3. Nondimensionalized center plate deflection and stress for $[0/90]_s$ laminate subjected to sinusoidal pressure; E_1/E_2=25, G_{12}/E_2=G_{13}/E_2=0.5, G_{23}/E_2=0.2, υ_{12}=0.25; M,N=1.

$S=s/h$	$\bar{w}$	$\bar{\sigma}_1$	$\bar{\sigma}_2$	$\bar{\sigma}_4$	$\bar{\sigma}_5$
100	0.4343	0.5387	0.2708	0.1117	0.2897
20	0.5060	0.5393	0.3043	0.1234	0.2825
10	0.7147	0.5456	0.3888	0.1531	0.2640
4	1.8937	0.6651	0.6322	0.2389	0.2064

CROSS-PLY CYLINDRICAL SHELL

Similar to the flat plate case, a $[0/90]_s$ cross-ply cylindrical shell subjected to transverse sinusoidal pressure is solved exactly by one term (M,N=1) in the assumed displacement of Eqns (22). The present solution is compared to a solution given by Reddy (1982) who assumed the lower-order Reissner-Mindlin transverse shear deformation theory. Table 4 compares the two approaches for a thick shell geometry, $S=s/h$=10 and two curvature ratios, R/h=10 or 100. The present Galerkin approach gives a more flexible response for both R/h ratios as expected due to its more accurate transverse shear representation. Also shown is the solution from a mesh of shear deformable finite elements by Palazotto and Dennis (1992). The finite element solution is also based on the

parabolic transverse shear distribution through the shell thickness. However, although the Galerkin and Reddy results are based on Donnell shallow shell theory, the finite element solution can model deep shell behavior. Even for the deeper shell (R/h=10), the present Donnell Galerkin equations give fairly accurate results as compared to the deep shell finite element solution.

The shell Batdorf-Stein parameter Z, defined in Eqn (30) for balanced symmetric laminates, is useful in shell studies where the response of different geometries are compared (Nemeth, 1994).

$$Z = \frac{s^2}{R}\left[\frac{A_{11}A_{22} - A_{12}^2}{12\sqrt{A_{11}A_{22}D_{11}D_{22}}}\right]^{1/2} \tag{30}$$

where s is the circumferential length of the shell, A_{ij} and D_{ij} are defined Eqn (9).

For comparison of like material unidirectional [0] and [90] shell laminates, Z reduces to Eqn (31). Eqn (31) has been used in nondimensional buckling and nonlinear laminated shell studies by Dennis et al (1993,1994). Shallow shells are indicated by small values of Z and the flat plate limiting case has Z=0. The values of Z for the shells of this section are shown in the Tables.

$$Z = \frac{s^2}{Rh}\left[1 - \upsilon_{12}\upsilon_{21}\right]^{1/2} \tag{31}$$

Convergence characteristics of the present Galerkin solution for a shell geometry are seen by subjecting the $[0/90]_s$ shell laminate to a uniform transverse pressure. Results are shown in Table 5 for odd *m,n* terms only and are compared to an unpublished finite element solution.

QUASI-ISOTROPIC PLATES AND SHELLS

Two quasi-isotropic geometries are subjected to a uniform load. The Galerkin solution is compared to the published results of Phan and Reddy (1985) in Table 6 for a $[0/\pm45/90]_s$ flat plate laminate using the nondimensionalization of Eqn (29). The Galerkin solution is based on including terms of both the even and odd *m,n* of Eqn (22) as the nonzero D_{16} and D_{26} stiffnesses due to the ±45° plies destroy the biaxial symmetry that exists in the previous cases. However, the Phan results assumed symmetry in their study and this accounts for the differences with the present where for every thickness ratio, the Galerkin solution gives a stiffer response. An independent finite element analysis based on meshes of the 28 degree of freedom flat plate element developed by Palazotto and Dennis (1992) confirmed this where center plate deflections of Table 6 were also found both with and without the symmetry assumed.

Secondly, deflection and stress are shown in Table 7 for a $[-60/0/60]_s$ quasi-isotropic shell laminate for several geometries. Again, the even *m,n* terms in the assumed

displacement are required due to the nonzero D_{16} and D_{26} terms. For a given value of the Batdorf-Stein parameter Z, the thicker shells represented by smaller values of S undergo a more flexible response. Transverse shear flexibility accounts for the larger deflections. For a given thickness ratio s/h, as the curvature is increased as indicated by increasing Z, both the center shell deflection and circumferential stress decrease due to more membrane action and less bending. These results are also consistent with the response given in Table 4. For many of the geometries with larger Z, the thicker shell responses (smaller S) are not shown as R/h becomes too small.

TABLE 4. Nondimensionalized center shell deflection and stress for cross-ply cylindrical shell subjected to sinusoidal transverse pressure; E_1/E_2=25, G_{12}/E_2=G_{13}/E_2=0.5, G_{23}/E_2=0.2, υ_{12}=0.25; S=s/h=r/h=10; M,N=1.

	Z=1.177 R/h=100			Z=11.77 R/h=10		
	$\bar{w}$	$\bar{\sigma}_1$	$\bar{\sigma}_2$	$\bar{w}$	$\bar{\sigma}_1$	$\bar{\sigma}_2$
Galerkin	0.71237	0.5501	0.3938	0.53645	0.4570	0.3393
Reddy (1982)	0.6609	0.4998	0.3637	0.5006	0.4233	0.3190
Finite Element*	0.71354	0.5520	0.3923	0.55665	0.4751	0.3075

*Finite element solution based an 8x8 quarter model mesh of 36 degree of freedom shell elements developed by Dennis and Palazotto (1992).

TABLE 5. Nondimensionalized center shell deflection and stress for cross-ply cylindrical shell subjected to uniform transverse pressure; E_1/E_2=25, G_{12}/E_2=G_{13}/E_2=0.5, G_{23}/E_2=0.2, υ_{12}=0.25; Z=11.77, R/h=1000, s/h=100.

	M, N: 1	5	9	13	Finite Element*
$\bar{w}$	0.5858	0.5663	0.5661	0.5661	0.5662
$\bar{\sigma}_1$	0.7785	0.7302	0.7286	0.7283	0.7293
$\bar{\sigma}_2$	0.4172	0.3378	0.3339	0.3332	0.3324

*Finite element solution based an 8x8 quarter model mesh of 36 degree of freedom shell elements developed by Dennis and Palazotto (1992).

TABLE 6. Nondimensionalized center plate deflection for $[0/\pm45/90]_s$ laminate subjected to uniform load; E_1/E_2=40, G_{12}/E_2=G_{13}/E_2=0.6, G_{23}/E_2=0.5, υ_{12}=0.25; m,n = 1,2,3,...M,N

$S=s/h$	M,N 1	3	9	15	*Phan (1985)
100	0.3380	0.3354	0.3372	0.3377	0.3769
50	0.3444	0.3412	0.3431	0.3437	0.3859
20	0.3893	0.3818	0.3847	0.3852	0.4336
10	0.5486	0.5259	0.5316	0.5321	0.5904
4	1.6275	1.5070	1.5283	1.5289	1.6340

*Assumed plate biaxial symmetry.

TABLE 7. Nondimensionalized center plate and shell deflection and stress for $[-60/0/60]_s$ laminate subjected to uniform load, E_1/E_2=40, G_{12}/E_2=G_{13}/E_2=0.6, G_{23}/E_2=0.5, υ_{12}=0.25; m,n = 1,2,3,...21. Stress is compressive except for asteriked value.

Z	$S=s/h$	R/h	$\bar{w}$	$\bar{\sigma}_2$ ($z=-h/2$)
0	100	-	0.3557	0.5149
	20	-	0.4006	0.5341
	10	-	0.5307	0.5072
	5	-	1.0142	0.5949
1.9384	100	5000	0.3461	0.4657
	20	200	0.3878	0.4908
	10	50	0.5042	0.5337
	5	12.5	0.9191	0.6223
4.846	100	2000	0.2948	0.4426
	20	80	0.3238	0.4583
	10	20	0.3987	0.4761
	5	5	-	-
9.692	100	1000	0.1916	0.3298
	20	40	0.2019	0.3272
	10	10	0.2245	0.3063
	5	2.5	-	-
24.230	100	400	0.05199	0.1062
48.461	100	200	0.01192	0.01800
96.922	100	100	0.0008239	*0.01061

ANGLE-PLY PLATE

A thick plate consisting of a single 45° ply is subjected to both a uniform and sinusoidal transverse pressure and nondimensional center plate deflection and stress are shown in Table 8. This laminate has nonzero A_{16} and A_{26} membrane stiffnesses and its effect apparently slows convergence considerably compared to other results of this section where the solution including 24 terms, both even and odd, is yet unconverged.

TABLE 8. Nondimensionalized center plate deflection and bending stress ($z=\pm h/2$) for angle-ply [45] $E_1/E_2=40$, $G_{12}/E_2=G_{13}/E_2=0.5$; $G_{23}/E_2=0.2$, $\upsilon_{12}=0.25$; $S=a/h=10$, $a=r=s$.

	M,N			
load	3	6	12	24
uniform				
$\bar{w}$	0.6632	0.7073	0.7569	0.7877
$\bar{\sigma}_1$	0.4609	0.3621	0.3885	0.4175
sinusoidal				
$\bar{w}$	0.4257	0.4499	0.4794	0.4969
$\bar{\sigma}_1$	0.3198	0.2552	0.2749	0.2909

V. Nonlinear Plate and Shell Solutions

Geometric nonlinearity becomes important for plate and shell cases where the transverse displacement is no longer small compared to the thickness. The assumed von Karman nonlinearity of the in-plane strains of Eqn (3) is valid when the deflection is of the same order of magnitude as the thickness of the shell. Consequently, the nonlinear solutions presented here will give the plate and shell response up to several shell thicknesses of deflection.

An iterative algorithm gives the solution to the nonlinear equilibrium equations of Appendix B, see Eqns (B1)-(B8). The algorithm uses direct iteration with residual forces determining convergence. Normally, if a norm of the displacement vector changed by less than 0.1% from the previous iteration, convergence was assumed.

Including the nonlinear terms of Eqns (B6)-(B8) requires numerous calculations for solution of the shell equilibrium path. To gain an appreciation for the additional computation burden of the nonlinear solution, compare the number of passes through loops of computer coding for the linear versus the nonlinear solution. Although the number of mathematical operations is different for a loop within the linear versus the nonlinear coding, counting the number of passes through loops will still illustrate the significant additional math required. Let n equal the number of terms in one of the summations in the assumed displacement of Eqn (22). Calculating all stiffnesses of the *linear* coefficient matrix of Eqns (B1)-(B5) requires n^4 passes through algorithm loops. The *nonlinear* coefficient matrix is a sum of the *linear* stiffnesses that are calculated only one time and the stiffnesses of Eqns (B6)-(B8). Calculation of the stiffnesses of Eqns (B6)-(B8) requires n^8 loops *for each iteration* of the nonlinear algorithm. Several iterations are typically required for convergence for each increment of load that represents a single point on the nonlinear equilibrium path. Since accurate solutions were typically found using nine terms in both x and y coordinate directions (for 81 terms) for each of the assumed displacements of Eqn (22), the nonlinear solution algorithm consumes significantly more computing time over the linear solution. In cases where biaxial symmetry exists, the solutions given resulted from m,n =1,3,5,...17 in the series of Eqn (22).

The nonlinear Galerkin solution technique is verified by calculating the response of an isotropic plate subjected to uniform pressure. The classical solution was published by S. Levy in 1942 and is a typical nonlinear plate comparison. Next, plate and cylindrical shell solutions are presented for unidirectional and cross-ply laminates. The effect of transverse shear deformation in the presence of geometric nonlinearity is discussed.

ISOTROPIC PLATE

A square isotropic plate is subjected to uniform transverse pressure and the response is plotted in Figure 2. The present response of Figure 2 resulted from m,n=1,3,5,...17 in the assumed displacement of Eqn (22). For increasing load, membrane resistance results in a stiffer plate response in the nonlinear solution compared to the linear

response where no membrane-bending coupling is modelled. Presently, the nonlinear response diverges from the linear solution for deflections larger than approximately one third of the plate thickness. Although the response shown compares well to the published results of Levy (1942) for the load range given, at higher loads the solution begins to diverge from the published and increasingly more terms in the assumed displacement are required for accuracy.

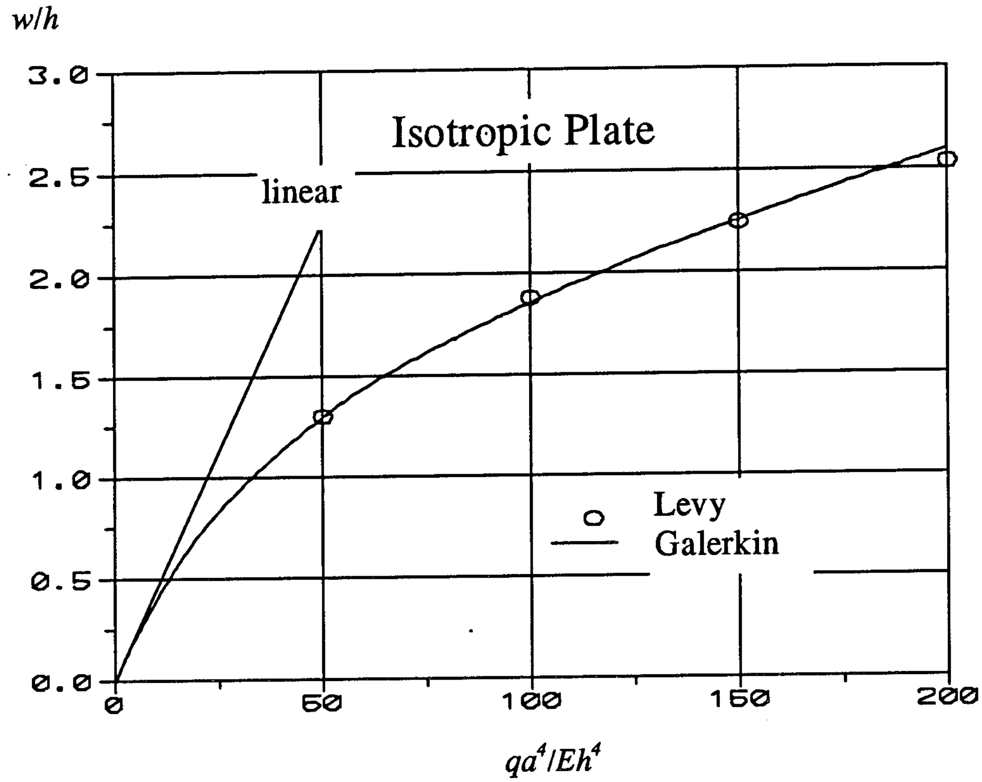

FIGURE 2. Center plate deflection versus load for a square (r=s=a) isotropic plate subjected to uniform pressure; υ=0.316.

UNIDIRECTIONAL PLATE LAMINATE

Next consider a plate constructed with fibers aligned the x coordinate, i.e., a [0] laminate, subjected to sinusoidal transverse loading of in Eqn (27). The response is shown in Figure 3 for two plate thickness ratios (S=a/h, r=s=a). The linear response for both plates is also shown in the figure. Due to the simple sinusoidal loading, a nearly converged response results by including only a few terms in the assumed displacement. In the previous uniform load case, both the load and the response must be approximated by the trigonometric series. In this case, the load is modelled exactly by only one term and hence easier convergence in the displacement response is seen.

The thick plate undergoes a more flexible response compared to the thin plate due to increased transverse shear flexibility. However, transverse shear deformation in the nonlinear analysis is less than in the linear analysis since in the nonlinear, membrane stiffening is a competing influence. For this case, whereas the linear response is 15.3% more flexible in the thick plate compared to the thin plate for all load levels, the nonlinear is only 13.4% greater for nondimensionalized load of 250, 11.1% greater for a load of 500, and only 10.6% greater for a nondimensionalized load of 750.

Figure 4 shows the center plate normal stress, σ_1, for the *thin* [0] laminate, linear versus nonlinear. The stress calculated at the plate center for the *thick* plate is approximately the same as that shown in Figure 4, the thin response being slightly higher than the thick plate response.

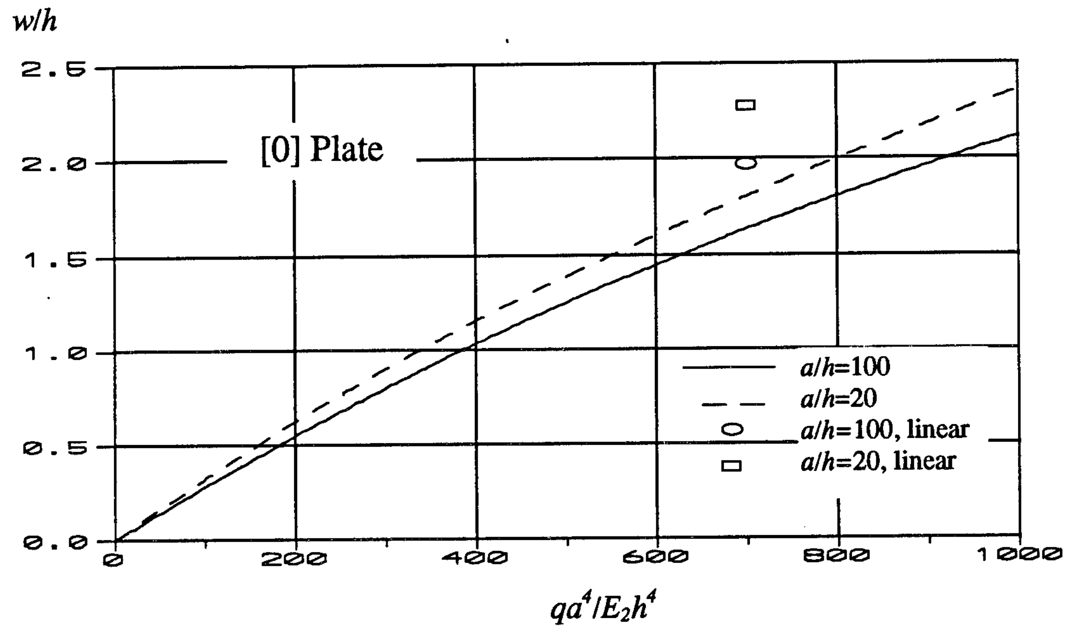

FIGURE 3. Center plate deflection versus load for a square ($r=s=a$) unidirectional laminate subjected to sinusoidal transverse pressure; $E_1/E_2=40$, $G_{12}/E_2=G_{13}/E_2=0.6$, $G_{23}/E_2=0.5$, $\upsilon_{12}=0.25$; $M,N=17$ (odd m,n only).

CROSS-PLY PLATE LAMINATE

The response of a $[0/90]_S$ laminate under uniform transverse pressure is given for two plate thickness ratios in Figure 5. As was true for the unidirectional plate, the thicker plates undergo a more flexible response compared to the thin plate due to transverse shear deformation. The flexibility due to transverse shear is less in the nonlinear response compared to the linear response as membrane stiffening is a competing influence similar to that seen in the unidirectional case.

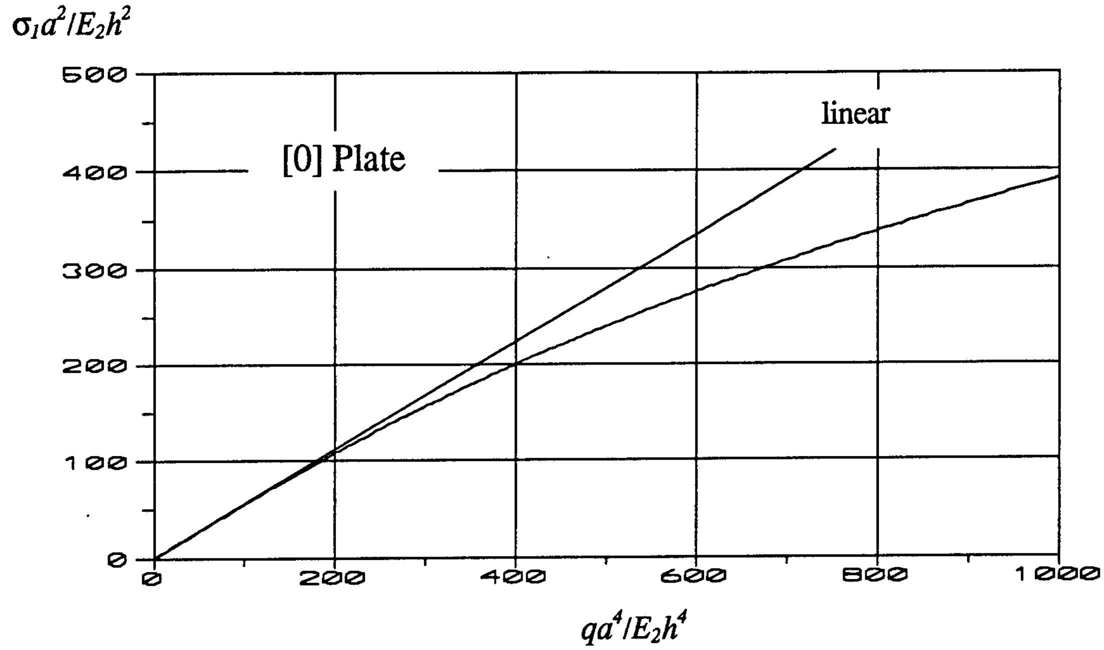

FIGURE 4. Center plate normal stress in unidirectional square (r=s=a) laminate subjected to sinusoidal transverse pressure; E_1/E_2=40, G_{12}/E_2=G_{13}/E_2=0.6, G_{23}/E_2=0.5, υ_{12}=0.25; S=a/h=100; M,N=17 (odd m,n only). Stress extreme tensile, z=h/2.

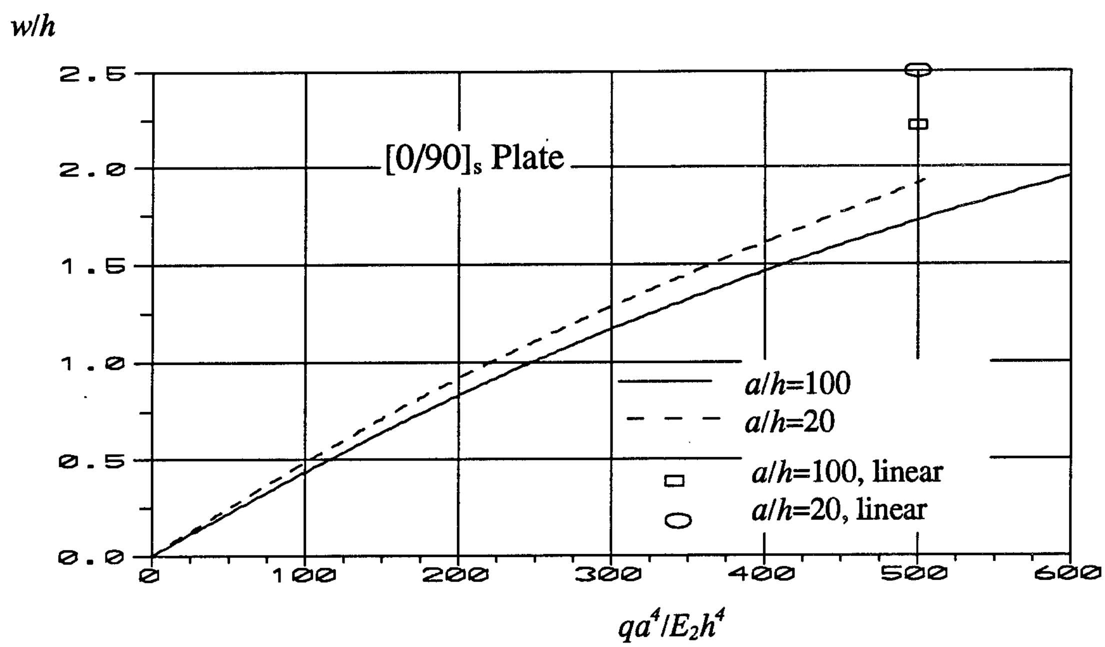

FIGURE 5. Center plate deflection versus load for a square (r=s=a) cross-ply plate subjected to uniform pressure; E_1/E_2=40, G_{12}/E_2=G_{13}/E_2=0.6, G_{23}/E_2=0.5, υ_{12}=0.25; M,N=17 (odd m,n only).

UNIDIRECTIONAL SHELL LAMINATE

Unidirectional [0] and [90] laminates are subjected to sinusoidal transverse pressure. Figure 6 gives the center shell response of a [90] laminate for the Batdorf-Stein parameter of Eqn (30), Z=9.992. Initially, the shell increasingly deflects with load due to the flexibility gained through circumferential membrane compression. However, the enlarging longitudinal tension field eventually overcomes the flexibility gained through membrane compression and the shell begins to stiffen with load. For all loads, the thicker shell (R/h=40, a/h=20) is more flexible. The response of a [0] laminate was also calculated and for this value of Z, the response was nearly identical to the [90].

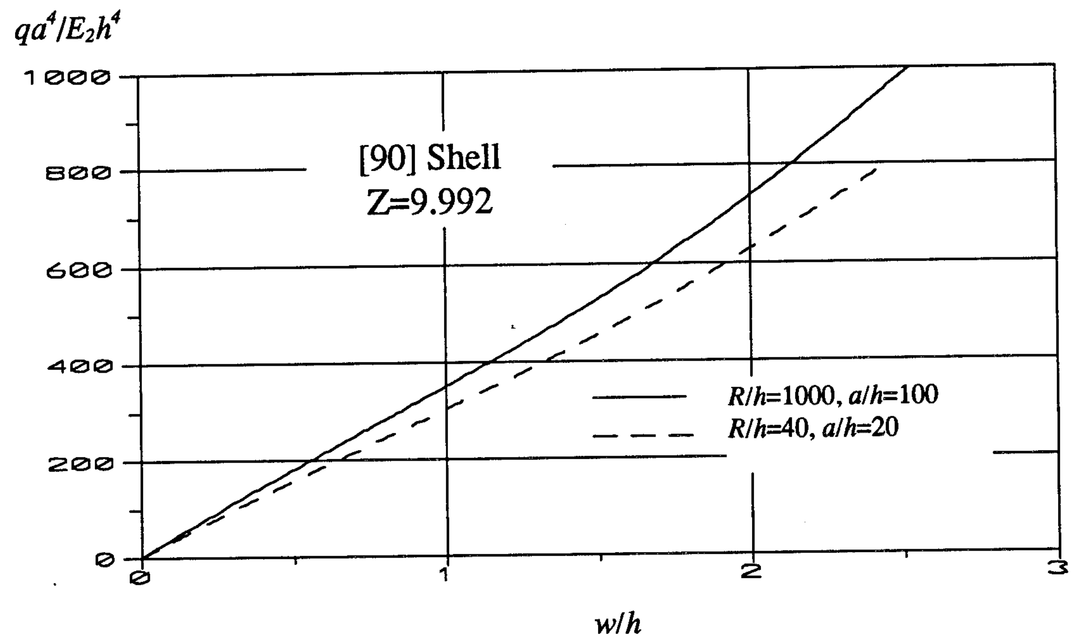

FIGURE 6. Center shell deflection versus load for a square (r=s=a) unidirectional shell laminate subjected to sinusiodal transverse pressure; E_1/E_2=40, G_{12}/E_2=G_{13}/E_2=0.6, G_{23}/E_2=0.5, υ_{12}=0.25; M,N=17 (odd m,n only).

CROSS-PLY CYLINDRICAL SHELL LAMINATE

A thin cross-ply shell is subjected to sinusoidal transverse pressure and compared to a finite element solution in Figure 7. Due to membrane compression, the shell initially exhibits more flexible behavior as the load is increased, i.e., the shell softens with load. Eventually, significant deflection results when the load is increased only a small amount as given by the shallow slope in the load-deflection curve. The longitudinal bending tension field on the bottom of the shell enlarges until its stiffening influence overtakes the flexibility caused by membrane compression and the shell stiffens for the highest loads shown in the figure. The Galerkin solution follows the finite element result well except the

latter reaches a slightly higher load before the very shallow slope in the load-deflection curve begins.

Figure 8 shows the nonlinear response of two cross-ply shells that have a Batdorf-Stein parameter equal to 11.945. One shell geometry is thin and shallow and should experience insignificant transverse shear deformation. The second shell is a deeper, thicker shell where the more flexible response shown in the figure is probably due to transverse shear deformation. Although the geometry of the shell (R, h, r, s) is the same as for the unidirectional shell of Figure 6, the Batdorf-Stein parameter of Eqn (30) is larger (9.992 versus 11.945) due to the ply stacking. Interestingly however, both thin and thick shell responses are nearly identical for the unidirectional and cross-ply laminates; and furthermore, have the same value Z if Eqn (31) were used where ply stacking is not considered in the parameter definition.

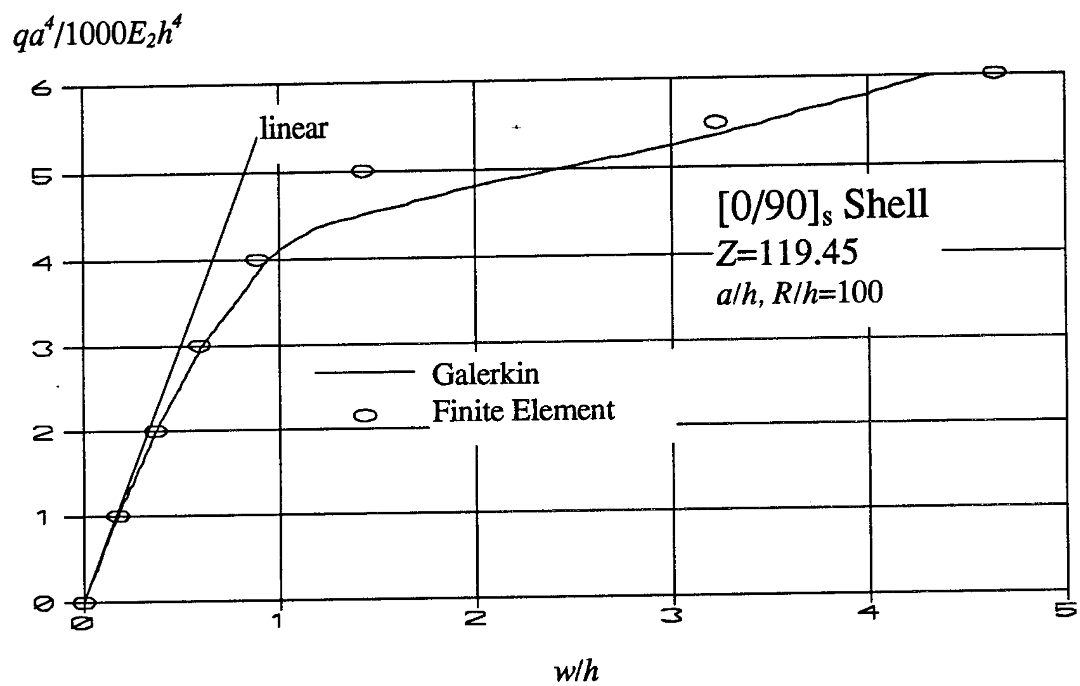

FIGURE 7. Center shell deflection versus load for a square (r=s=a) cross-ply shell laminate subjected to sinusiodal transverse pressure; E_1/E_2=40, G_{12}/E_2=G_{13}/E_2=0.6, G_{23}/E_2=0.5, υ_{12}=0.25; M,N=17 (odd m,n only).

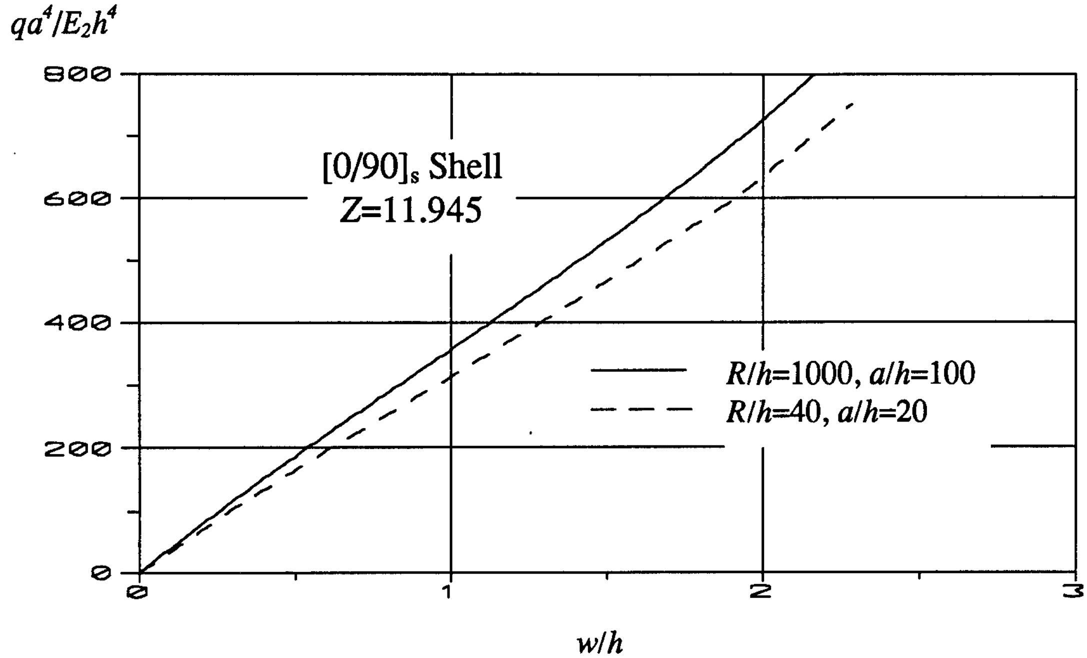

FIGURE 8. Center shell deflection versus load for square (r=s=a) cross-ply shell laminates subjected to sinusiodal transverse pressure; E_1/E_2=40, G_{12}/E_2=G_{13}/E_2=0.6, G_{23}/E_2=0.5, υ_{12}=0.25; M,N=17 (odd m,n only).

VI. Conclusions

A laminated shallow shell approach that includes von Karman geometric nonlinearity and parabolic transverse shear deformation has been posed in differential operator form. Trigonometric functions are then assumed for each of the five shell displacement degrees of freedom for the subsequent nonlinear galerkin solution. The galerkin nonlinear solution is computationally intensive. The response of several laminate geometries subjected to transverse loadings are found. Thicker plates and shells generally exhibit more flexible response compared to thinner geometries in both linear and nonlinear analyses. The nondimensional shell response is examined by using the Batdorf-Stein shell parameter for laminated constructions. Quasi-isotropic shallow shells undergo significant transverse shear flexibility in the thicker geometries as given by the nondimensional shell crown deflection. However, the nondimensional crown deflection in the deeper shell response is not much influenced by shell thickness. For flat plates, geometric nonlinearity lessens the influence of transverse shear flexibility when compared to linear solutions due to membrane stretching resistance.

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APPENDIX A
TRIGONOMETRIC INTEGRALS

Evaluation of the trigonometric integrals resulting from the assumed displacement of Eqn (22) substituted into Eqn (23) makes use of the trigonometric identities of Eqn(A1).

$$\begin{aligned}\cos A\cos B &= \frac{1}{2}[\cos(A-B)+\cos(A+B)]\\ \sin A\sin B &= \frac{1}{2}[\cos(A-B)-\cos(A+B)]\\ \sin A\cos B &= \frac{1}{2}[\sin(A-B)+\sin(A+B)]\end{aligned} \tag{A1}$$

Linear stiffness terms require evaluation of the three integrals of Eqns (A2)-(A4):
Note: *i, j, k, l,* are integers and

$$x' = \frac{\pi x}{r}$$

$$\int_0^r \cos ix' \cos jx'\, dx = \begin{cases} \frac{r}{2} & if \quad i = j \\ 0 & if \quad i \neq j \end{cases} \tag{A2}$$

$$\int_0^r \sin ix' \sin jx'\, dx = \begin{cases} \frac{r}{2} & if \quad i = j \\ 0 & if \quad i \neq j \end{cases} \tag{A3}$$

$$\int_0^r \sin ix' \cos jx'\, dx = \begin{cases} \frac{2ri}{\pi(i^2 - j^2)} & if \quad i + j = odd \\ 0 & if \quad i + j = even \end{cases} \tag{A4}$$

Nonlinear stiffness terms require evaluation of the four integrals of Eqns (A5)-(A8).

$$\int_0^r \sin ix' \sin jx' \sin kx'\, dx = \frac{r}{2\pi}\left\{\frac{1}{t_1} + \frac{1}{t_2} - \frac{1}{t_3} - \frac{1}{t_4}\right\} \tag{A5}$$

$$\int_0^r \cos ix' \cos jx' \sin kx' \, dx = \frac{r}{2\pi}\left\{\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right\} \tag{A6}$$

where,

$$t_1 = \begin{cases} k-(i-j) & \text{if} \quad k-(i-j)=odd \\ 0 & \text{if} \quad k-(i-j)=even \end{cases}$$

$$t_2 = \begin{cases} k+(i-j) & \text{if} \quad k+(i-j)=odd \\ 0 & \text{if} \quad k+(i-j)=even \end{cases}$$

$$t_3 = \begin{cases} k-(i+j) & \text{if} \quad k-(i+j)=odd \\ 0 & \text{if} \quad k-(i+j)=even \end{cases}$$

$$t_4 = \begin{cases} k+(i+j) & \text{if} \quad k+(i+j)=odd \\ 0 & \text{if} \quad k+(i+j)=even \end{cases}$$

$$\int_0^r \sin ix' \sin jx' \sin kx' \sin lx' \, dx = \frac{r}{8}\left(n_1+n_2-n_3-n_4-n_5-n_6+n_7\right) \tag{A8}$$

$$\int_0^r \cos ix' \cos jx' \sin kx' \sin lx' \, dx = \frac{r}{8}\left(n_1+n_2-n_3-n_4+n_5+n_6-n_7\right) \tag{A9}$$

where,

$$n_1 = \begin{cases} 1 & \text{if} \quad (i-j)-(k-l)=0 \\ 0 & \text{if} \quad otherwise \end{cases}$$

$$n_2 = \begin{cases} 1 & \text{if} \quad (i-j)+(k-l)=0 \\ 0 & \text{if} \quad otherwise \end{cases}$$

$$n_3 = \begin{cases} 1 & \text{if} \quad (i-j)-(k+l)=0 \\ 0 & \text{if} \quad otherwise \end{cases}$$

$$n_4 = \begin{cases} 1 & \text{if} \quad (i-j)+(k+l)=0 \\ 0 & \text{if} \quad otherwise \end{cases}$$

$$n_5 = \begin{cases} 1 & \text{if} \quad (i+j)-(k-l)=0 \\ 0 & \text{if} \quad otherwise \end{cases}$$

$$n_6 = \begin{cases} 1 & \text{if} \quad (i+j)+(k-l)=0 \\ 0 & \text{if} \quad otherwise \end{cases}$$

$$n_7 = \begin{cases} 1 & \text{if} \quad (i+j)-(k+l)=0 \\ 0 & \text{if} \quad otherwise \end{cases}$$

APPENDIX B
GALERKIN EQUATIONS

Eqns (23) are given below in terms of constants a_{ij}, b_{ij}, c_{ij}, d_{ij}, and e_{ij}. Five equilibrium equations, (B1)-(B5), result for every p,q of Eqn (23). The summations extend to M,N see Eqn (22).

Linear terms

Note: i, j, p, q are positive integers ranging from 1 to $M=N$, k_h of Eqns (B1)-(B5) is the parameter defined in Eqn (1), shell planform rxs.

From Eqn (23a):

$$\sum_i \sum_j (f_3 f_6)\left(-\frac{2A_{16}}{rs} ij\pi^2 a_{ij} - \frac{A_{16}}{r^2} i^2\pi^2 b_{ij} - \frac{A_{26}}{s^2} j^2\pi^2 b_{ij} - \frac{A_{26}}{sR} j\pi c_{ij}\right) +$$
$$(f_1 f_2)\left(-\frac{A_{11}}{r^2} i^2\pi^2 a_{ij} - \frac{A_{66}}{s^2} j^2\pi^2 a_{ij} - \frac{A_{12}+A_{66}}{rs} ij\pi^2 b_{ij} - \frac{A_{12}}{rR} i\pi c_{ij}\right) + \tag{B1}$$
$$f_6\left(\frac{A_{16}}{s} j\pi a_{ij} + \frac{A_{16}}{r} i\pi b_{ij}\right)(1-\cos i\pi \cos p\pi) = 0$$

From Eqn (23b):

$$\sum_i \sum_j (f_4 f_5)\left(-\frac{2A_{26}}{rs} ij\pi^2 b_{ij} - \frac{A_{16}}{r^2} i^2\pi^2 a_{ij} - \frac{A_{26}}{s^2} j^2\pi^2 a_{ij} - \frac{A_{26}}{rR} i\pi c_{ij}\right) +$$
$$(f_1 f_2)\left(-\frac{A_{66}}{r^2} i^2\pi^2 b_{ij} - \frac{A_{22}}{s^2} j^2\pi^2 b_{ij} - \frac{A_{12}+A_{66}}{rs} ij\pi^2 a_{ij} - \frac{A_{22}}{sR} j\pi c_{ij}\right) + \tag{B2}$$
$$f_4\left(\frac{A_{26}}{s} j\pi a_{ij} + \frac{A_{26}}{r} i\pi b_{ij}\right)(1-\cos j\pi \cos q\pi) = 0$$

From Eqn (23c):

$$
\begin{aligned}
&\sum_i \sum_j (f_4 f_6)\Big(-\frac{A_{26}}{Rs} j\pi a_{ij} - \frac{A_{26}}{rR} i\pi b_{ij} - \frac{4H_{26}}{rs^3} k_h^2 i j^3 \pi^4 c_{ij} - \frac{4H_{16}}{r^3 s} k_h^2 i^3 j \pi^4 c_{ij} - \frac{2A_{45}}{rs} ij\pi^2 c_{ij} \\
&- \frac{12D_{45}}{rs} k_h ij\pi^2 c_{ij} - \frac{18F_{45}}{rs} k_h^2 ij\pi^2 c_{ij} - \frac{F_{26}}{s^3} k_h j^3 \pi^3 d_{ij} - \frac{H_{26}}{s^3} k_h^2 j^3 \pi^3 d_{ij} \\
&- \frac{A_{45}}{s} j\pi d_{ij} - \frac{6D_{45}}{s} k_h j\pi d_{ij} - \frac{9F_{45}}{s} k_h^2 j\pi d_{ij} - \frac{3F_{16}}{r^2 s} k_h i^2 j\pi^3 d_{ij} - \frac{3H_{16}}{r^2 s} k_h^2 i^2 j\pi^3 d_{ij} \\
&- \frac{F_{16}}{r^3} k_h i^3 \pi^3 e_{ij} - \frac{H_{16}}{r^3} k_h^2 i^3 \pi^3 e_{ij} - \frac{A_{45}}{r} i\pi e_{ij} - \frac{6D_{45}}{r} k_h i\pi e_{ij} - \frac{9F_{45}}{r} k_h^2 i\pi e_{ij} \\
&- \frac{3F_{26}}{rs^2} k_h i j^2 \pi^3 e_{ij} - \frac{3H_{26}}{rs^2} k_h^2 i j^2 \pi^3 e_{ij} \Big) + \\
&(f_1 f_2)\Big(\frac{A_{12}}{rR} i\pi a_{ij} + \frac{A_{22}}{sR} j\pi b_{ij} + \frac{A_{22}}{R^2} c_{ij} + \frac{H_{11}}{r^4} k_h^2 i^4 \pi^4 c_{ij} + \frac{A_{55}}{r^2} i^2 \pi^2 c_{ij} + \frac{6D_{55}}{r^2} k_h i^2 \pi^2 c_{ij} \\
&+ \frac{9F_{55}}{r^2} k_h^2 i^2 \pi^2 c_{ij} + \frac{H_{22}}{s^4} k_h^2 j^4 \pi^4 c_{ij} + \frac{A_{44}}{s^2} j^2 \pi^2 c_{ij} + \frac{6D_{44}}{s^2} k_h j^2 \pi^2 c_{ij} + \frac{9F_{44}}{s^2} k_h^2 j^2 \pi^2 c_{ij} \\
&+ \frac{2(H_{12} + 2H_{66})}{r^2 s^2} k_h^2 i^2 j^2 \pi^4 c_{ij} + \frac{F_{11}}{r^3} k_h i^3 \pi^3 d_{ij} + \frac{H_{11}}{r^3} k_h^2 i^3 \pi^3 d_{ij} + \frac{A_{55}}{r} i\pi d_{ij} \\
&+ \frac{6D_{55}}{r} k_h i\pi d_{ij} + \frac{9F_{55}}{r} k_h^2 i\pi d_{ij} + \frac{F_{12}}{rs^2} i j^2 \pi^3 d_{ij} + \frac{2F_{66}}{rs^2} k_h i j^2 \pi^3 d_{ij} + \frac{H_{12}}{rs^2} k_h^2 i j^2 \pi^3 d_{ij} \Big) + \\
&(f_6)\left(\frac{k_h \pi}{r} p \right)\left(\frac{2H_{16}}{rs} k_h ij\pi^2 c_{ij} + \frac{F_{16}}{s} j\pi d_{ij} + \frac{H_{16}}{s} k_h j\pi d_{ij} + \frac{F_{16}}{r} i\pi e_{ij} + \frac{H_{16}}{r} k_h i\pi e_{ij} \right)(\cos i\pi \cos p\pi - 1) + \\
&(f_4)\left(\frac{k_h \pi}{s} q \right)\left(\frac{2H_{26}}{rs} k_h ij\pi^2 c_{ij} + \frac{F_{26}}{s} j\pi d_{ij} + \frac{H_{26}}{s} k_h j\pi d_{ij} + \frac{F_{26}}{r} i\pi e_{ij} + \frac{H_{26}}{r} k_h i\pi e_{ij} \right)(\cos j\pi \cos q\pi - 1) \\
&\quad = load
\end{aligned}
\tag{B3}
$$

From Eqn (23d):

$$\sum_i \sum_j (f_3 f_6)\Big(-\frac{F_{26}}{s^3}k_h j^3\pi^3 c_{ij} - \frac{H_{26}}{s^3}k_h^{\,2} j^3\pi^3 c_{ij} - \frac{A_{45}}{s} j\pi c_{ij} - \frac{6D_{45}}{s}k_h j\pi c_{ij} - \frac{9F_{45}}{s}k_h^{\,2} j\pi c_{ij}$$

$$-\frac{3F_{16}}{r^2 s}k_h i^2 j\pi^3 c_{ij} - \frac{3H_{16}}{r^2 s}k_h^{\,2} i^2 j\pi^3 c_{ij} - \frac{2D_{16}}{rs} ij\pi^2 d_{ij} - \frac{4F_{16}}{rs}k_h ij\pi^2 d_{ij} - \frac{2H_{16}}{rs}k_h^{\,2} ij\pi^2 d_{ij}$$

$$-A_{45}e_{ij} - 6D_{45}k_h e_{ij} - 6F_{45}k_h^{\,2} e_{ij} - \frac{D_{16}}{r^2} i^2\pi^2 e_{ij} - \frac{2F_{16}}{r^2}k_h i^2\pi^2 e_{ij} - \frac{H_{16}}{r^2}k_h^{\,2} i^2\pi^2 e_{ij}$$

$$-\frac{D_{26}}{s^2} j^2\pi^2 e_{ij} - \frac{2F_{26}}{s^2}k_h j^2\pi^2 e_{ij} - \frac{H_{26}}{s^2}k_h^{\,2} j^2\pi^2 e_{ij}\Big) +$$

$$(f_1 f_2)\Big(-\frac{F_{11}}{r^3}k_h i^3\pi^3 c_{ij} - \frac{H_{11}}{r^3}k_h^{\,2} i^3\pi^3 c_{ij} - \frac{A_{55}}{r} i\pi c_{ij} - \frac{6D_{55}}{r}k_h i\pi c_{ij} - \frac{9F_{55}}{r}k_h^{\,2} i\pi c_{ij} - \frac{F_{12}}{rs^2}k_h ij^2\pi^3 c_{ij}$$

$$-\frac{2F_{66}}{rs^2}k_h ij^2\pi^3 c_{ij} - \frac{H_{12}}{rs^2}k_h^{\,2} ij^2\pi^3 c_{ij} - \frac{2H_{66}}{rs^2}k_h^{\,2} ij^2\pi^3 c_{ij} - A_{55}d_{ij} - 6D_{55}k_h d_{ij} - 9F_{55}k_h^{\,2} d_{ij}$$

$$-\frac{D_{11}}{r^2} i^2\pi^2 d_{ij} - \frac{2F_{11}}{r^2}k_h i^2\pi^2 d_{ij} - \frac{H_{11}}{r^2}k_h^{\,2} i^2\pi^2 d_{ij} - \frac{D_{66}}{s^2} j^2\pi^2 d_{ij} - \frac{2F_{66}}{s^2}k_h j^2\pi^2 d_{ij}$$

$$-\frac{H_{66}}{s^2}k_h^{\,2} j^2\pi^2 d_{ij} - \frac{D_{12}}{rs} ij\pi^2 e_{ij} - \frac{D_{66}}{rs} ij\pi^2 e_{ij} - \frac{2F_{12}}{rs}k_h ij\pi^2 e_{ij} - \frac{2F_{66}}{rs}k_h ij\pi^2 e_{ij} - \frac{H_{12}}{rs}k_h^{\,2} ij\pi^2 e_{ij}$$

$$-\frac{H_{66}}{rs}k_h^{\,2} ij\pi^2 e_{ij}\Big) +$$

$$(f_6)\left(\frac{2F_{16}}{rs} ij\pi^2 c_{ij} + \frac{D_{16}}{s} j\pi d_{ij} + \frac{F_{16}}{s}k_h j\pi d_{ij} + \frac{D_{16}}{r} i\pi e_{ij} + \frac{F_{16}}{r}k_h i\pi e_{ij}\right)(1-\cos i\pi \cos p\pi) +$$

$$(f_6)\left(\frac{2H_{16}}{rs}k_h ij\pi^2 c_{ij} + \frac{F_{16}}{s} j\pi d_{ij} + \frac{H_{16}}{s}k_h j\pi d_{ij} + \frac{F_{16}}{r} i\pi e_{ij} + \frac{H_{16}}{r}k_h i\pi e_{ij}\right)k_h(1-\cos i\pi \cos p\pi) = 0$$

(B4)

From Eqn (23e):

$$
\begin{aligned}
&\sum_i \sum_j (f_1 f_2)\Big(-\frac{F_{22}}{s^3}k_h j^3\pi^3 c_{ij} - -\frac{H_{22}}{s^3}k_h^2 j^3\pi^3 c_{ij} - \frac{A_{44}}{s} j\pi c_{ij} - -\frac{6D_{44}}{s}k_h j\pi c_{ij} - \frac{9F_{44}}{s}k_h^2 j\pi c_{ij} \\
&-\frac{F_{12}}{r^2 s}k_h i^2 j\pi^3 c_{ij} - \frac{2F_{66}}{r^2 s}k_h i^2 j\pi^3 c_{ij} - \frac{H_{12}}{r^2 s}k_h^2 i^2 j\pi^3 c_{ij} - \frac{2H_{66}}{r^2 s}k_h^2 i^2 j\pi^3 c_{ij} - \frac{D_{12}}{rs} ij\pi^2 d_{ij} \\
&-\frac{D_{66}}{rs} ij\pi^2 d_{ij} - \frac{2F_{12}}{rs}k_h ij\pi^2 d_{ij} - \frac{2F_{66}}{rs}k_h ij\pi^2 d_{ij} - \frac{H_{12}}{rs}k_h^2 ij\pi^2 d_{ij} - \frac{H_{66}}{rs}k_h^2 ij\pi^2 d_{ij} \\
&-A_{44}e_{ij} - 6D_{44}ke_{ij} - 9F_{44}k^2 e_{ij} - \frac{D_{66}}{r^2} i^2\pi^2 e_{ij} - \frac{2F_{66}}{r^2} k i^2\pi^2 e_{ij} - \frac{H_{66}}{r^2} k^2 i^2\pi^2 e_{ij} \\
&-\frac{D_{22}}{s^2} j^2\pi^2 e_{ij} - \frac{2F_{22}}{s^2}k_h j^2\pi^2 e_{ij} - \frac{H_{22}}{s^2}k_h^2 j^2\pi^2 e_{ij}\Big) + \\
&(f_4 f_5)\Big(-\frac{F_{16}}{r^3}k_h i^3\pi^3 c_{ij} - \frac{H_{16}}{r^3}k_h^2 i^3\pi^3 c_{ij} - \frac{A_{45}}{r} i\pi c_{ij} - \frac{6D_{45}}{r}k_h i\pi c_{ij} - \frac{9F_{45}}{r}k_h^2 i\pi c_{ij} \\
&-\frac{3F_{26}}{rs^2}k_h ij^2\pi^3 c_{ij} - \frac{3H_{26}}{rs^2}k_h^2 ij^2\pi^3 c_{ij} - A_{45}d_{ij} - 6D_{45}k_h d_{ij} - 9F_{45}k_h^2 d_{ij} \\
&-\frac{D_{16}}{r^2} i^2\pi^2 d_{ij} - \frac{2F_{16}}{r^2}k_h i^2\pi^2 d_{ij} - \frac{H_{16}}{r^2}k_h^2 i^2\pi^2 d_{ij} - \frac{D_{26}}{s^2} j^2\pi^2 d_{ij} - \frac{2F_{26}}{s^2}k_h j^2\pi^2 d_{ij} \\
&-\frac{H_{26}}{s^2}k_h^2 j^2\pi^2 d_{ij} - \frac{2D_{26}}{rs} ij\pi^2 e_{ij} - \frac{4F_{26}}{rs}k_h ij\pi^2 e_{ij} - \frac{2H_{26}}{rs}k_h^2 ij\pi^2 e_{ij}\Big) + \\
&(f_4)\left(\frac{2F_{26}}{rs}k_h ij\pi^2 c_{ij} + \frac{D_{26}}{s} j\pi d_{ij} + \frac{F_{26}}{s}k_h j\pi d_{ij} + \frac{D_{26}}{r} i\pi e_{ij} + \frac{F_{26}}{r}k_h i\pi e_{ij}\right)(1-\cos j\pi \cos q\pi) + \\
&(f_4)\left(\frac{2H_{26}}{rs}k_h ij\pi^2 c_{ij} + \frac{F_{26}}{s} j\pi d_{ij} + \frac{H_{26}}{s}k_h j\pi d_{ij} + \frac{F_{26}}{r} i\pi e_{ij} + \frac{H_{26}}{r}k_h i\pi e_{ij}\right)k_h(1-\cos j\pi \cos q\pi) = 0
\end{aligned}
$$

(B5)

where in Eqns (B1)-(B5):

$$
f_1 = \begin{cases} \dfrac{r}{2} & if \quad i = p \\[2ex] 0 & if \quad i \neq p \end{cases}
$$

$$
f_2 = \begin{cases} \dfrac{s}{2} & if \quad j = q \\[2ex] 0 & if \quad j \neq q \end{cases}
$$

$$f_3 = \begin{cases} \dfrac{2ri}{\pi(i^2 - p^2)} & if \quad i + p = odd \\ 0 & if \quad i + p = even \end{cases}$$

$$f_4 = \begin{cases} \dfrac{2rp}{\pi(p^2 - i^2)} & if \quad i + p = odd \\ 0 & if \quad i + p = even \end{cases}$$

$$f_5 = \begin{cases} \dfrac{2sj}{\pi(j^2 - q^2)} & if \quad j + q = odd \\ 0 & if \quad j + q = even \end{cases}$$

$$f_6 = \begin{cases} \dfrac{2sq}{\pi(q^2 - j^2)} & if \quad j + q = odd \\ 0 & if \quad j + q = even \end{cases}$$

Nonlinear terms

The additional nonlinear terms below in Eqns (B6)-(B8) are included on the left hand side of Eqns (B1)-(B3). The nonlinear A_{16} and A_{26} terms are not shown and i, j, k, l, m, n, p, q are positive integers ranging from 1 to $M=N$.

From Eqn (23a):

$$\sum_i\sum_j\sum_k\sum_l c_{ij}c_{kl}\Bigg(-A_{11}\left(\frac{i\pi}{r}\right)\left(\frac{k\pi}{r}\right)^2\frac{r}{2\pi}\left(\frac{1}{x_1}-\frac{1}{x_2}-\frac{1}{x_3}+\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)+$$
$$(A_{12}+A_{66})\left(\frac{j\pi}{s}\right)\left(\frac{kl\pi^2}{rs}\right)\frac{r}{2\pi}\left(-\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}+\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right)-$$
$$A_{66}\left(\frac{j\pi}{s}\right)^2\left(\frac{k\pi}{r}\right)\frac{r}{2\pi}\left(-\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}+\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)+$$
$$\frac{A_{11}}{2}\left(\frac{i\pi}{r}\right)\left(\frac{k\pi}{r}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)(1-\cos i\pi\cos k\pi\cos p\pi)\Bigg)$$

(B6)

From Eqn (23b):

$$\sum_i\sum_j\sum_k\sum_l c_{ij}c_{kl}\Bigg(-A_{22}\left(\frac{j\pi}{s}\right)\left(\frac{l\pi}{s}\right)^2\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}-\frac{1}{y_2}-\frac{1}{y_3}+\frac{1}{y_4}\right)+$$
$$(A_{12}+A_{66})\left(\frac{i\pi}{r}\right)\left(\frac{kl\pi^2}{rs}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)\frac{s}{2\pi}\left(-\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}+\frac{1}{y_4}\right)-$$
$$A_{66}\left(\frac{i\pi}{r}\right)^2\left(\frac{l\pi}{s}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(-\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}+\frac{1}{y_4}\right)+$$
$$\frac{A_{22}}{2}\left(\frac{j\pi}{s}\right)\left(\frac{l\pi}{s}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)(1-\cos j\pi\cos l\pi\cos q\pi)\Bigg)$$

(B7)

From Eqn (23c):

$$\sum_i\sum_j\sum_k\sum_l a_{ij}c_{kl}\Big(-A_{12}\left(\frac{i\pi}{r}\right)\left(\frac{l\pi}{s}\right)^2\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)-$$

$$2A_{66}\left(\frac{j\pi}{s}\right)\left(\frac{kl\pi^2}{rs}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right)-$$

$$A_{11}\left(\frac{k\pi}{r}\right)^2\left(\frac{i\pi}{r}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)\Big)+$$

$$\sum_i\sum_j\sum_k\sum_l b_{ij}c_{kl}\Big(-A_{12}\left(\frac{j\pi}{s}\right)\left(\frac{k\pi}{r}\right)^2\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)-$$

$$2A_{66}\left(\frac{i\pi}{r}\right)\left(\frac{kl\pi^2}{rs}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right)-$$

$$A_{22}\left(\frac{l\pi}{s}\right)^2\left(\frac{j\pi}{s}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)\Big)+$$

$$\sum_i\sum_j\sum_k\sum_l c_{ij}c_{kl}\Big(-\frac{A_{22}}{2R}\left(\frac{j\pi}{s}\right)\left(\frac{l\pi}{s}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right)-$$

$$\frac{A_{12}}{R}\left(\frac{k\pi}{r}\right)^2\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)-$$

$$\frac{A_{12}}{2R}\left(\frac{k\pi}{r}\right)\left(\frac{i\pi}{r}\right)\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)-$$

$$\frac{A_{22}}{R}\left(\frac{l\pi}{s}\right)^2\frac{r}{2\pi}\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_3}-\frac{1}{x_4}\right)\frac{s}{2\pi}\left(\frac{1}{y_1}+\frac{1}{y_2}-\frac{1}{y_3}-\frac{1}{y_4}\right)\Big)+$$

$$\sum_i\sum_j\sum_k\sum_l\sum_m\sum_n c_{ij}c_{kl}c_{mn}\cdot$$

$$\left(\frac{A_{22}}{2}\left(\frac{j\pi}{s}\right)\left(\frac{l\pi}{s}\right)\left(\frac{n\pi}{s}\right)^2(g_1-g_2-g_3+g_4+g_5-g_6-g_7)(h_1-h_2+h_3-h_4+h_5-h_6+h_7)+\right.$$

$$\frac{A_{12}}{2}\left(\frac{j\pi}{s}\right)^2\left(\frac{k\pi}{r}\right)\left(\frac{m\pi}{r}\right)(g_1-g_2-g_3+g_4-g_5+g_6+g_7)(h_1-h_2-h_3+h_4+h_5-h_6-h_7)-$$

$$A_{66}\left(\frac{i\pi}{r}\right)\left(\frac{l\pi}{s}\right)\left(\frac{mn\pi^2}{rs}\right)(g_1-g_2-g_3+g_4-g_5+g_6+g_7)(h_1-h_2-h_3+h_4-h_5+h_6+h_7)-$$

$$A_{66}\left(\frac{k\pi}{r}\right)\left(\frac{j\pi}{s}\right)\left(\frac{mn\pi^2}{rs}\right)(g_1-g_2-g_3+g_4-g_5+g_6+g_7)(h_1-h_2-h_3+h_4-h_5+h_6+h_7)+$$

$$\frac{A_{11}}{2}\left(\frac{k\pi}{r}\right)\left(\frac{i\pi}{r}\right)\left(\frac{m\pi}{r}\right)^2(g_1-g_2+g_3-g_4+g_5-g_6+g_7)(h_1-h_2-h_3+h_4+h_5-h_6-h_7)+$$

$$\left.\frac{A_{12}}{2}\left(\frac{i\pi}{r}\right)^2\left(\frac{l\pi}{s}\right)\left(\frac{n\pi}{s}\right)(g_1-g_2-g_3+g_4+g_5-g_6-g_7)(h_1-h_2-h_3+h_4-h_5+h_6+h_7)\right) \tag{B8}$$

where x_i and y_i are defined similar to the t_i of Appendix A. Instead of indices k,i,j for the t_i, the x_i are given by substituting p,i,k respectively. The y_i are defined likewise by substituting indices q, j, l respectively. The g_i and the h_i are defined from the n_i of Appendix A. Instead of indices i,j,k,l for the n_i, the g_i are given by substituting i,k,m,p respectively. The h_i are defined likewise by substituting indices j,l,n,q respectively.

APPENDIX C
FORTRAN LISTING
SAMPLE INPUT/OUTPUT LISTINGS

```
C 21mar95, galerkin.FOR/even and odd terms
C GALERKIN SOLUTION TO DONNELL SHALLOW SHELL, LAMINATED MATERIAL
C PARABOLIC TRANSVERSE SHEAR residual force
C
C to change to odd terms only, look for C $$$$$
C change to i=1,nterms*2-1,2
C*****************************************************************
C INPUT:
C
C NANAL(1) (ISO=1, LAM=2), NANAL(2) (LINEAR=1,  NONLINEAR =2)
C FOR NONLINEAR: NINC,IMAX
C NTERMS, count only odd terms
C NPRINT
C NLOAD(SIN=0, UNIFORM=1, CENTER PT LOAD=2), MAG LOAD=QO
C EY,NU,HT *OR* E1,E2,G12,NU12,G13,G23
C FOR LAM: NP,HT
C TH1,TH2,TH3...NP
C R,S,RAD (IF FLAT PLATE, INPUT RAD=0.)
C
C PANEL RXS, RADIUS=RAD
C*****************************************************************
C
C ARRAYS SIZED FOR 21 TERMS-LINEAR AND 15 TERMS-NONLINEAR
   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
   CHARACTER*64 FNAME,GNAME
   COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
  X    STNL13(225,225),STNL23(225,225),STNL31(225,225),
  X    STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
   COMMON/ELAS/NANAL(3),E1,E2,G12,NU12,PT,THE(20),NP,A(3,3),
  X        B(3,3),D(3,3),E(3,3),F(3,3),H(3,3),
  X        AS(2,2),DS(2,2),FS(2,2),G13,
  X        G23,EY,NU,HT,RTHE(20),GS
   COMMON/SOLVE/INDX(2205),VV(2205)
   COMMON/CONVERGE/DIS(2205),DISPREV(2205)
   COMMON/STRS/Q11TOP,Q12TOP,Q22TOP,Q16TOP,Q26TOP,
  X        QS11,QS22,Q12,Q22
   DOUBLE PRECISION K1,NU,NU12,NU21
   WRITE(*,923)
   READ(*,925)FNAME
   WRITE(*,924)
   READ(*,925)GNAME
 923 FORMAT(' WHAT IS YOUR INPUT FILE NAME?')
 924 FORMAT(' WHAT IS YOUR OUTPUT FILE NAME?')
 925 FORMAT (A)
   OPEN(5,FILE=FNAME)
   OPEN(6,FILE=GNAME)
   ITER=0
   KTINC=1
   CALL INPUT(NPRNT,NINC,IMAX)
   CALL ELAST(NPRNT,S,PHO)
   CALL STIFFNESS
  20 CALL NLST(ITER)
   CALL SUBST(NANAL(2),ITER)
   CALL LUDCMP
   CALL LUBKSB
   CALL CONVRGE(NCV,ITER,IMAX,KTINC)
   CALL DISPLACEMENT(NTERMS,NCV,HT,PHO,R,S,NANAL(2))
   IF(KTINC.LE.NINC .AND. NANAL(2) .EQ. 2)GOTO 20
   END
C 19oct94
C*****************************************************************
C
   SUBROUTINE STIFFNESS
C
C*****************************************************************
C...............................................................
C   THIS SUBROUTINE FILLS LINEAR STIFFNESS MATRIX
C            Kd=f
C...............................................................
   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
   COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
```

```
  X    STNL13(225,225),STNL23(225,225),STNL31(225,225),
  X    STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
   COMMON/ELAS/NANAL(3),E1,E2,G12,NU12,PT,THE(20),NP,A(3,3),
  X        B(3,3),D(3,3),E(3,3),F(3,3),H(3,3),
  X        AS(2,2),DS(2,2),FS(2,2),G13,
  X        G23,EY,NU,HT,RTHE(20),GS
C
   DOUBLE PRECISION K1,NU,NU12,NU21
   PI=3.14159265359
   PI4=PI*PI*PI*PI
   K1=-4./(3.*HT**2)
   M=1
   N=1
   DO 100 II=1,5*NTERMS*NTERMS
   DO 200 JJ=1,5*NTERMS*NTERMS
 200 ST(II,JJ)=0.
 100 PLOAD(II)=0.
C $$$$$
   DO 300 LP=1,NTERMS
   DO 300 LQ=1,NTERMS
   DO 350 I=1,NTERMS
   DO 350 J=1,NTERMS
C
C SET UP PARAMETERS F FOR INDICES M,LP
C
   ITEST1=MOD(I+LP,2)
   ITEST2=MOD(J+LQ,2)
   IF (I.EQ.LP) THEN
    F1=R/2.
    F3=0.
    F4=0.
   ELSE
    F1=0.
    IF (ITEST1.EQ.0) THEN
     F3=0.
     F4=0.
    ELSE
     F3=(2.*I*R)/(PI*(I*I-LP*LP))
     F4=(2.*LP*R)/(PI*(LP*LP-I*I))
    ENDIF
   ENDIF
C
C SET UP PARAMETERS FOR INDICES N,LQ
C
   IF (J.EQ.LQ) THEN
    F2=S/2.
    F5=0.
    F6=0.
   ELSE
    F2=0.
    IF (ITEST2.EQ.0) THEN
     F5=0.
     F6=0.
    ELSE
     F5=(2.*J*S)/(PI*(J*J-LQ*LQ))
     F6=(2.*LQ*S)/(PI*(LQ*LQ-J*J))
    ENDIF
   ENDIF
C
C FILL IN STIFFNESS, ST, FOR BC-1
C
C EQN 1
C
    A1 = F3*F6*2.*A(1,3)*I*J*PI*PI/(R*S)+
   XF1*F2*(A(1,1)*I*I*PI*PI/(R*R) +
   XA(3,3)*J*J*PI*PI/(S*S)) -
   XF6*A(1,3)*J*PI/S*(1.-DCOS(I*PI)*DCOS(LP*PI))
C
    B1 = F3*F6*(A(1,3)*I*I*PI*PI/(R*R) +
   XA(2,3)*J*J*PI*PI/(S*S))+
   XF1*F2*(A(1,2)+A(3,3))*I*J*PI*PI/(R*S)-
```

```
     XF6*A(1,3)*I*PI/R*(1.-DCOS(I*PI)*DCOS(LP*PI))
C
      C1 = F3*F6*A(2,3)*J*PI*PHO/S +
     XF1*F2*A(1,2)*I*PI*PHO/R
C
C EQN 2
C
      A2 = F1*F2*(A(1,2)+A(3,3))*I*J*PI*PI/(R*S)+
     XF4*F5*(A(1,3)*I*I*PI*PI/(R*R)+
     XA(2,3)*J*J*PI*PI/(S*S))-
     XF4*A(2,3)*J*PI/S*(1.-DCOS(J*PI)*DCOS(LQ*PI))
C
      B2 =F1*F2*(A(3,3)*I*I*PI*PI/(R*R)+
     XA(2,2)*J*J*PI*PI/(S*S))+
     XF4*F5*2.*A(2,3)*I*J*PI*PI/(R*S)-
     XF4*A(2,3)*I*PI/R*(1.-DCOS(J*PI)*DCOS(LQ*PI))
C
      C2 =F1*F2*A(2,2)*J*PI*PHO/S +
     XF4*F5*A(2,3)*I*PI*PHO/R
C
C EQN 3
C
      A3 = F4*F6*A(2,3)*J*PI*PHO/S -
     XF1*F2*A(1,2)*I*PI*PHO/R
C
      B3 = F4*F6*A(2,3)*I*PI*PHO/R -
     XF1*F2*A(2,2)*J*PI*PHO/S
C
      C3 = F4*F6*(4.*H(2,3)*K1*K1*PI4*I*J**3/(R*S**3)+
     X4.*H(1,3)*K1*K1*I**3*J*PI4/(R**3*S)+
     X2.*AS(1,2)*I*J*PI*PI/(R*S)+
     X12.*DS(1,2)*K1*I*J*PI*PI/(R*S)+
     X18.*FS(1,2)*K1*K1*I*J*PI*PI/(R*S))+
     XF1*F2*(-A(2,2)*PHO*PHO - (H(1,1)*K1*K1*I**4*PI4/R**4 +
     XAS(2,2)*I*I*PI*PI/(R*R)+
     X6.*DS(2,2)*K1*I*I*PI*PI/(R*R)+
     X9.*FS(2,2)*K1*K1*I*I*PI*PI/(R*R)+
     XH(2,2)*K1*K1*J**4*PI4/(S**4)+
     XAS(1,1)*J*J*PI*PI/(S*S)+
     X6.*DS(1,1)*K1*J*J*PI*PI/(S*S)+
     X9.*FS(1,1)*K1*K1*J*J*PI*PI/(S*S)+
     X2.*(H(1,2)+2.*H(3,3))*K1*K1*I*I*J*J*PI4/(R*R*S*S)))-
     XF6*(K1*LP*PI/R)*2.*H(1,3)*K1*I*J/(R*S)*PI*PI*
     X(DCOS(I*PI)*DCOS(LP*PI)-1.)-
     XF4*(K1*LQ*PI/S)*2.*H(2,3)*K1*I*J/(R*S)*PI*PI*
     X(DCOS(J*PI)*DCOS(LQ*PI)-1.)
C
      D3 = F4*F6*(F(2,3)*K1*J**3*PI*PI*PI/(S**3)+
     XH(2,3)*K1*K1*J**3*PI*PI*PI/(S**3)+
     XAS(1,2)*J*PI/S+6.*DS(1,2)*K1*J*PI/S+
     X9.*FS(1,2)*K1*K1*J*PI/S+
     X3.*F(1,3)*K1*I*I*J*PI*PI*PI/(R*R*S)+
     X3.*H(1,3)*K1*K1*I*I*J*PI*PI*PI/(R*R*S))-
     XF1*F2*(F(1,1)*K1*I**3*PI*PI*PI/(R**3)+
     XH(1,1)*K1*K1*I**3*PI*PI*PI/(R**3)+
     XAS(2,2)*I*PI/R+6.*DS(2,2)*K1*I*PI/R+
     X9.*FS(2,2)*K1*K1*I*PI/R+
     XF(1,2)*K1*I*J*J*PI*PI*PI/(R*S*S)+
     X2.*F(3,3)*K1*I*J*J*PI*PI*PI/(R*S*S)+
     XH(1,2)*K1*K1*I*J*J*PI*PI*PI/(R*S*S)+
     X2.*H(3,3)*K1*K1*I*J*J*PI*PI*PI/(R*S*S))-
     XF6*(K1*LP*PI/R)*(F(1,3)*J/S*PI+H(1,3)*K1*J*PI/S)*
     X(DCOS(I*PI)*DCOS(LP*PI)-1.)-
     XF4*(K1*LQ*PI/S)*(F(2,3)*J/S*PI+H(2,3)*K1*J*PI/S)*
     X(DCOS(J*PI)*DCOS(LQ*PI)-1.)
C
      E3 = F4*F6*(F(1,3)*K1*I**3*PI*PI*PI/(R**3)+
     XH(1,3)*K1*K1*I**3*PI*PI*PI/(R**3)+
     XAS(1,2)*I*PI/R+6.*DS(1,2)*K1*I*PI/R+
     X9.*FS(1,2)*K1*K1*I*PI/R+
     X3.*F(2,3)*K1*I*J*J*PI*PI*PI/(R*S*S)+
```

```
     X3.*H(2,3)*K1*K1*I*J*PI*PI*PI/(R*S*S))-
     XF1*F2*(F(2,2)*K1*J*J*J*PI*PI*PI/(S**3)+
     XH(2,2)*K1*K1*J**3*PI**3/S**3+
     XAS(1,1)*J*PI/S+6.*DS(1,1)*K1*J*PI/S+
     X9.*FS(1,1)*K1*K1*J*PI/S+
     XF(1,2)*K1*I*I*J*PI**3/(R*R*S)+
     X2.*F(3,3)*K1*I*I*J*PI**3/(R*R*S)+
     XH(1,2)*K1*K1*I*I*J*PI**3/(R*R*S)+
     X2.*H(3,3)*K1*K1*I*I*J*PI**3/(R*R*S))-
     XF6*(K1*LP*PI/R)*(F(1,3)*I/R*PI+H(1,3)*K1*I*PI/R)*
     X(DCOS(I*PI)*DCOS(LP*PI)-1.)-
     XF4*(K1*LQ*PI/S)*(F(2,3)*I/R*PI+H(2,3)*K1*I*PI/R)*
     X(DCOS(J*PI)*DCOS(LQ*PI)-1.)
C
C EQN 4
C
      C4 = F3*F6*(F(2,3)*K1*J**3*PI*PI*PI/(S**3) +
     XH(2,3)*K1*K1*J**3*PI*PI*PI/(S**3)+
     XAS(1,2)*J*PI/S+6.*DS(1,2)*K1*J*PI/S +
     X9.*FS(1,2)*K1*K1*J*PI/S+
     X3.*F(1,3)*K1*I*I*J*PI*PI*PI/(R*R*S)+
     X3.*H(1,3)*K1*K1*I*I*J*PI*PI*PI/(R*R*S))+
     XF1*F2*(F(1,1)*K1*I**3*PI*PI*PI/(R**3)+
     XH(1,1)*K1*K1*I**3*PI*PI*PI/(R**3)+
     XAS(2,2)*I*PI/R+6.*DS(2,2)*K1*I*PI/R +
     X9.*FS(2,2)*K1*K1*I*PI/R+
     XF(1,2)*K1*I*J*J*PI*PI*PI/(R*S*S)+
     X2.*F(3,3)*K1*I*J*J*PI*PI*PI/(R*S*S)+
     XH(1,2)*K1*K1*I*J*J*PI*PI*PI/(R*S*S)+
     X2.*H(3,3)*K1*K1*I*J*J*PI*PI*PI/(R*S*S))-
     XF6*2.*F(1,3)*I*J*K1*PI*PI/(R*S)*
     X(1.-DCOS(I*PI)*DCOS(LP*PI))-
     XF6*K1*2.*H(1,3)*K1*I*J*PI*PI/(R*S)*
     X(1.-DCOS(I*PI)*DCOS(LP*PI))
C
      D4 = F3*F6*(2.*D(1,3)*I*J*PI*PI/(R*S)+
     X4.*F(1,3)*K1*I*J*PI*PI/(R*S)+
     X2.*H(1,3)*K1*K1*I*J*PI*PI/(R*S))+
     XF1*F2*(AS(2,2)+6.*K1*DS(2,2)+9.*FS(2,2)*K1*K1+
     XD(1,1)*I*I*PI*PI/(R*R)+
     X2.*F(1,1)*K1*I*I*PI*PI/(R*R)+
     XH(1,1)*K1*K1*I*I*PI*PI/(R*R)+
     XD(3,3)*J*J*PI*PI/(S*S)+
     X2.*F(3,3)*K1*J*J*PI*PI/(S*S)+
     XH(3,3)*K1*K1*J*J*PI*PI/(S*S))-
     XF6*(D(1,3)*J*PI/S+F(1,3)*K1*J*PI/S)*
     X(1.-DCOS(I*PI)*DCOS(LP*PI))-
     XF6*K1*(F(1,3)*J*PI/S+H(1,3)*K1*J*PI/S)*
     X(1.-DCOS(I*PI)*DCOS(LP*PI))
C
      E4 = F3*F6*(AS(1,2) +6.*DS(1,2)*K1+
     X9.*FS(1,2)*K1*K1+D(1,3)*I*I*PI*PI/(R*R)+
     X2.*F(1,3)*K1*I*I*PI*PI/(R*R)+
     XH(1,3)*K1*K1*I*I*PI*PI/(R*R)+
     XD(2,3)*J*J*PI*PI/(S*S)+
     X2.*F(2,3)*K1*J*J*PI*PI/(S*S)+
     XH(2,3)*K1*K1*J*J*PI*PI/(S*S))+
     XF1*F2*(D(1,2)*I*J*PI*PI/(R*S)+
     XD(3,3)*I*J*PI*PI/(R*S)+
     X2.*F(1,2)*K1*I*J*PI*PI/(R*S)+
     X2.*F(3,3)*K1*I*J*PI*PI/(R*S)+
     XH(1,2)*K1*K1*I*J*PI*PI/(R*S)+
     XH(3,3)*K1*K1*I*J*PI*PI/(R*S))-
     XF6*(D(1,3)*I*PI/R+F(1,3)*K1*I*PI/R)*
     X(1.-DCOS(I*PI)*DCOS(LP*PI))-
     XF6*K1*(F(1,3)*I*PI/R+H(1,3)*K1*I*PI/R)*
     X(1.-DCOS(I*PI)*DCOS(LP*PI))
C
C EQN 5
C
      C5 =F1*F2*(F(2,2)*K1*J*J*J*PI*PI*PI/(S**3)+
```

```
     XH(2,2)*K1*K1*J**3*PI*PI*PI/(S**3)+
     XAS(1,1)*J*PI/S+6.*DS(1,1)*K1*J*PI/S+
     X9.*FS(1,1)*K1*K1*J*PI/S+
     XF(1,2)*K1*I*I*J*PI*PI*PI/(R*R*S)+
     X2.*F(3,3)*K1*I*I*J*PI*PI*PI/(R*R*S)+
     XH(1,2)*K1*K1*I*I*J*PI*PI*PI/(R*R*S)+
     X2.*H(3,3)*K1*K1*I*I*J*PI*PI*PI/(R*R*S))+
     XF4*F5*(F(1,3)*K1*I*I*I*PI*PI*PI/(R**3)+
     XH(1,3)*K1*K1*I*I*I*PI*PI*PI/(R**3)+
     XAS(1,2)*I*PI/R+6.*DS(1,2)*K1*I*PI/R+
     X9.*FS(1,2)*K1*K1*I*PI/R+
     X3.*F(2,3)*K1*I*J*J*PI*PI*PI/(R*S*S)+
     X3.*H(2,3)*K1*K1*I*J*J*PI*PI*PI/(R*S*S))-
     XF4*2.*F(2,3)*K1*I*J*PI*PI/(R*S)*
     X(1.-DCOS(J*PI)*DCOS(LQ*PI))-
     XF4*K1*2.*H(2,3)*K1*I*J*PI*PI/(R*S)*
     X(1.-DCOS(J*PI)*DCOS(LQ*PI))
C
      D5 =F1*F2*(D(1,2)*I*J*PI*PI/(R*S)+
     XD(3,3)*I*J*PI*PI/(R*S)+
     X2.*F(1,2)*K1*I*J*PI*PI/(R*S)+
     X2.*F(3,3)*K1*I*J*PI*PI/(R*S)+
     XH(1,2)*K1*K1*I*J*PI*PI/(R*S)+
     XH(3,3)*K1*K1*I*J*PI*PI/(R*S))+
     XF4*F5*(AS(1,2)+6.*K1*DS(1,2)+9.*K1*K1*FS(1,2)+
     XD(1,3)*I*I*PI*PI/(R*R)+
     X2.*F(1,3)*K1*I*I*PI*PI/(R*R)+
     XH(1,3)*K1*K1*I*I*PI*PI/(R*R)+
     XD(2,3)*J*J*PI*PI/(S*S)+
     X2.*F(2,3)*K1*J*J*PI*PI/(S*S)+
     XH(2,3)*K1*K1*J*J*PI*PI/(S*S))-
     XF4*(D(2,3)*J*PI/S+F(2,3)*K1*J*PI/S)*
     X(1.-DCOS(J*PI)*DCOS(LQ*PI))-
     XF4*K1*(F(2,3)*J*PI/S+H(2,3)*K1*J*PI/S)*
     X(1.-DCOS(J*PI)*DCOS(LQ*PI))

C
      E5 = F1*F2*(AS(1,1)+6.*DS(1,1)*K1+
     X9.*FS(1,1)*K1*K1+
     XD(3,3)*I*I*PI*PI/(R*R)+
     X2.*F(3,3)*K1*I*I*PI*PI/(R*R)+
     XH(3,3)*K1*K1*I*I*PI*PI/(R*R)+
     XD(2,2)*J*J*PI*PI/(S*S)+
     X2.*F(2,2)*K1*J*J*PI*PI/(S*S)+
     XH(2,2)*K1*K1*J*J*PI*PI/(S*S))+
     XF4*F5*(4.*F(2,3)*K1*I*J*PI*PI/(R*S)+
     X2.*H(2,3)*K1*K1*I*J*PI*PI/(R*S)+
     X2.*D(2,3)*I*J*PI*PI/(R*S))-
     XF4*(D(2,3)*I*PI/R+F(2,3)*K1*I*PI/R)*
     X(1.-DCOS(J*PI)*DCOS(LQ*PI))-
     XF4*K1*(F(2,3)*I*PI/R+H(2,3)*K1*I*PI/R)*
     X(1.-DCOS(J*PI)*DCOS(LQ*PI))
C
      ST(M,N) = -A1
      ST(M,N+NTERMS*NTERMS) = -B1
      ST(M,N+2*NTERMS*NTERMS) = -C1
C
      ST(M+NTERMS*NTERMS,N) = -A2
      ST(M+NTERMS*NTERMS,N+NTERMS*NTERMS) = -B2
      ST(M+NTERMS*NTERMS,N+2*NTERMS*NTERMS) = -C2
C
      ST(M+2*NTERMS*NTERMS,N)= -A3
      ST(M+2*NTERMS*NTERMS,N+NTERMS*NTERMS) = -B3
      ST(M+2*NTERMS*NTERMS,N+2*NTERMS*NTERMS) = -C3
      ST(M+2*NTERMS*NTERMS,N+3*NTERMS*NTERMS)= -D3
      ST(M+2*NTERMS*NTERMS,N+4*NTERMS*NTERMS)= -E3
C
      ST(M+3*NTERMS*NTERMS,N+2*NTERMS*NTERMS)= -C4
      ST(M+3*NTERMS*NTERMS,N+3*NTERMS*NTERMS)= -D4
      ST(M+3*NTERMS*NTERMS,N+4*NTERMS*NTERMS)= -E4
C
```

```
      ST(M+4*NTERMS*NTERMS,N+2*NTERMS*NTERMS)= -C5
      ST(M+4*NTERMS*NTERMS,N+3*NTERMS*NTERMS)= -D5
      ST(M+4*NTERMS*NTERMS,N+4*NTERMS*NTERMS)= -E5
C
C LOAD VECTOR FOR UNIFORM PRESSURE LOADING
C
      IF(I.EQ.LP .AND. J.EQ.LQ .AND. NLOAD.EQ.1)
     XPLOAD(M+2*NTERMS*NTERMS)=F1*F2*16*Q0/(PI*PI*I*J)
C
C LOAD VECTOR FOR CENTER PANEL POINT LOAD
C
      IF(I.EQ.LP .AND. J.EQ.LQ .AND. NLOAD.EQ.2)
     XPLOAD(M+2*NTERMS*NTERMS)=Q0*DSIN(I*PI/2)*DSIN(J*PI/2)
      N=N+1
  350 CONTINUE
      M=M+1
      N=1
  300 CONTINUE
C
C LOAD VECTOR FOR SINUSOIDAL LOADING
C
      IF(NLOAD.EQ.0)PLOAD(1+2*NTERMS*NTERMS)=R*S*Q0/4.
C
      RETURN
      END
C
C 3dec 94
C*****************************************************************
C
      SUBROUTINE NLST(ITER)
C
C NONLINEAR STIFFNESS TERMS--NO A16, A26 TERMS
C
C*****************************************************************
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
     X   STNL13(225,225),STNL23(225,225),STNL31(225,225),
     X   STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
      COMMON/ELAS/NANAL(3),E1,E2,G12,NU12,PT,THE(20),NP,A(3,3),
     X        B(3,3),D(3,3),E(3,3),F(3,3),H(3,3),
     X        AS(2,2),DS(2,2),FS(2,2),G13,
     X        G23,EY,NU,HT,RTHE(20),GS
      COMMON/CONVERGE/DIS(2205),DISPREV(2205)
      DOUBLE PRECISION K1,NU,NU12,NU21
      IF(NANAL(2).EQ.1)RETURN
      IF(ITER.EQ.0)RETURN
      PI=3.14159265359
      MM=1
      NN=1
C $$$$$
      DO 10 LP=1,NTERMS
      IF(NTERMS.GT.9)PRINT*,LP
      DO 10 LQ=1,NTERMS
      DO 30 I=1,NTERMS
      DO 30 J=1,NTERMS
      C1=0.
      C2=0.
      A3=0.
      B3=0.
      C3=0.
      KT1=1
C $$$$$
      DO 50 K=1,NTERMS
      DO 50 L=1,NTERMS
C
C SET UP PARAMETERS X,Y FOR INDICES
C
      I1=MOD(LP-(I-K),2)
      I2=MOD(LP+(I-K),2)
      I3=MOD(LP-(I+K),2)
      I4=MOD(K+I+LP,2)
```

```
      IF (I1.EQ.0) THEN
       X1=0.
      ELSE
       X1=1./(LP-(I-K))
      ENDIF
      IF (I2.EQ.0) THEN
       X2=0.
      ELSE
       X2=1./(LP+(I-K))
      ENDIF
      IF (I3.EQ.0) THEN
       X3=0.
      ELSE
       X3=1./(LP-(I+K))
      ENDIF
      IF (I4.EQ.0) THEN
       X4=0.
      ELSE
       X4=1./(LP+(I+K))
      ENDIF
C
      J1=MOD(LQ-(J-L),2)
      J2=MOD(LQ+(J-L),2)
      J3=MOD(LQ-(J+L),2)
      J4=MOD(LQ+J+L,2)
      IF (J1.EQ.0) THEN
       Y1=0.
      ELSE
       Y1=1./(LQ-(J-L))
      ENDIF
      IF (J2.EQ.0) THEN
       Y2=0.
      ELSE
       Y2=1./(LQ+(J-L))
      ENDIF
      IF (J3.EQ.0) THEN
       Y3=0.
      ELSE
       Y3=1./(LQ-(J+L))
      ENDIF
      IF (J4.EQ.0) THEN
       Y4=0.
      ELSE
       Y4=1./(LQ+J+L)
      ENDIF
C
C EQN 1
C
      C11=-A(1,1)*I*PI/R*(K*PI/R)**2*R*S/(4*PI*PI)*
     X(X1-X3-X2+X4)*(Y1+Y2-Y3-Y4)
      C12=(A(1,2)+A(3,3))*J*PI/S*K/R*L/S*PI*PI*R*S/(4*PI*PI)*
     X(X2-X3-X1+X4)*(Y1+Y2+Y3+Y4)
      C13=-A(3,3)*(J*PI/S)**2*K*PI/R*R*S/(4*PI*PI)*
     X(X2-X3-X1+X4)*(Y1+Y2-Y3-Y4)
      C14=.5*A(1,1)*I*PI/R*K*PI/R*S/(2*PI)*(1.-DCOS(I*PI)*
     XDCOS(K*PI)*DCOS(LP*PI))*(Y1+Y2-Y3-Y4)
      SUM1=(C11+C12+C13+C14)*DIS(KT1+2*NTERMS*NTERMS)
      C1=C1+SUM1
C
C EQN 2
C
      C21=-A(2,2)*J*PI/S*(L*PI/S)**2*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(-Y2-Y3+Y1+Y4)
      C22=(A(1,2)+A(3,3))*I*PI/R*K/R*L/S*PI*PI*R*S/(4*PI*PI)*
     X(X1+X2+X3+X4)*(-Y1-Y3+Y2+Y4)
C
      C23=-A(3,3)*L*PI/S*(I*PI/R)**2*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y2-Y3-Y1+Y4)
      C24=.5*A(2,2)*J*PI/S*L*PI/S*R/(2*PI)*(1.-DCOS(J*PI)*
     XDCOS(L*PI)*DCOS(LQ*PI))*(X1+X2-X3-X4)
      SUM2=(C21+C22+C23+C24)*DIS(KT1+2*NTERMS*NTERMS)
```

```
      C2=C2+SUM2
C
C EQN 3
C
      A31=-A(1,2)*(L*PI/S)**2*I*PI/R*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y1+Y2-Y3-Y4)
      A34=-2*A(3,3)*K/R*L/S*PI*PI*J*PI/S*R*S/(4*PI*PI)*
     X(X1+X2+X3+X4)*(Y1+Y2+Y3+Y4)
      A36=-A(1,1)*(K*PI/R)**2*I*PI/R*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y1+Y2-Y3-Y4)
      SUM3=(A31+A34+A36)*DIS(KT1+2*NTERMS*NTERMS)
      A3=A3+SUM3
C
      B31=-A(1,2)*(K*PI/R)**2*J*PI/S*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y1+Y2-Y3-Y4)
      B34=-2*A(3,3)*K/R*L/S*PI*PI*I*PI/R*R*S/(4*PI*PI)*
     X(X1+X2+X3+X4)*(Y1+Y2+Y3+Y4)
      B36=-A(2,2)*(L*PI/S)**2*J*PI/S*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y1+Y2-Y3-Y4)
      SUM4=(B31+B34+B36)*DIS(KT1+2*NTERMS*NTERMS)
      B3=B3+SUM4
C
      C31=-A(2,2)*PHO/2*J*PI/S*L*PI/S*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y1+Y2+Y3+Y4)
      C32=-A(1,2)*PHO*(K*PI/R)**2*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y1+Y2-Y3-Y4)
      C33=-A(1,2)*PHO/2*I*PI/R*K*PI/R*R*S/(4*PI*PI)*
     X(X1+X2+X3+X4)*(Y1+Y2-Y3-Y4)
      C39=-A(2,2)*PHO*(L*PI/S)**2*R*S/(4*PI*PI)*
     X(X1+X2-X3-X4)*(Y1+Y2-Y3-Y4)
      SUM5=(C31+C32+C33+C39)*DIS(KT1+2*NTERMS*NTERMS)
C
      KT2=1
      SUM=0.
C $$$$$
      DO 70 M=1,NTERMS
      DO 70 N=1,NTERMS
C
      XX1=0.
      XX2=0.
      XX3=0.
      XX4=0.
      XX5=0.
      XX6=0.
      XX7=0.
C
      IF((I-K)-(M-LP).EQ.0) XX1=R/8
      IF((I-K)-(M+LP).EQ.0) XX4=R/8
      IF((I+K)-(M-LP).EQ.0) XX6=R/8
      IF((I+K)-(M+LP).EQ.0) XX3=R/8
      IF((I-K)+(M-LP).EQ.0) XX2=R/8
      IF((I-K)+(M+LP).EQ.0) XX5=R/8
      IF((I+K)+(M-LP).EQ.0) XX7=R/8
C
      YY1=0.
      YY2=0.
      YY3=0.
      YY4=0.
      YY5=0.
      YY6=0.
      YY7=0.
C
      IF((J-L)-(N-LQ).EQ.0) YY1=S/8
      IF((J-L)-(N+LQ).EQ.0) YY5=S/8
      IF((J+L)-(N-LQ).EQ.0) YY3=S/8
      IF((J+L)-(N+LQ).EQ.0) YY7=S/8
      IF((J-L)+(N-LQ).EQ.0) YY2=S/8
      IF((J-L)+(N+LQ).EQ.0) YY6=S/8
      IF((J+L)+(N-LQ).EQ.0) YY4=S/8
C
      C34=0.5*A(2,2)*J*PI/S*L*PI/S*(N*PI/S)**2*
```

```
 X(XX1+XX2+XX3-XX4-XX5-XX6-XX7)*(YY1+YY2+YY3+YY4-YY5-YY6-YY7)
  C35=(A(1,2)/2)*(J*PI/S)**2*K*PI/R*M*PI/R*
 X(XX1-XX2+XX3-XX4+XX5-XX6+XX7)*(YY1+YY2-YY3-YY4-YY5-YY6+YY7)
  C36A=-(A(3,3))*I*PI/R*L*PI/S*M/R*N/S*PI*PI*
 X(XX1-XX2+XX3-XX4+XX5-XX6+XX7)*(YY1-YY2-YY3+YY4-YY5+YY6+YY7)
  C36B=-(A(3,3))*K*PI/R*J*PI/S*M/R*N/S*PI*PI*
 X(XX1-XX2+XX3-XX4+XX5-XX6+XX7)*(YY1-YY2-YY3+YY4-YY5+YY6+YY7)
  C37=(A(1,2)/2)*(I*PI/R)**2*(L*PI/S)*N*PI/S*
 X(XX1+XX2+XX3-XX4-XX5-XX6-XX7)*(YY1-YY2-YY3+YY4-YY5+YY6+YY7)
  C38=0.5*A(1,1)*I*PI/R*K*PI/R*(M*PI/R)**2*
 X(XX1+XX2-XX3-XX4-XX5+XX6+XX7)*(YY1+YY2-YY3-YY4-YY5-YY6+YY7)
  SUM=(C34+C35+C36A+C36B+C37+C38)*DIS(KT2+2*NTERMS*NTERMS)+SUM
  KT2=KT2+1
70 CONTINUE
  SUM6=SUM*DIS(KT1+2*NTERMS*NTERMS)
  C3=C3+SUM5+SUM6
  KT1=KT1+1
50 CONTINUE
C
  STNL13(MM,NN) = C1
  STNL23(MM,NN) = C2
  STNL31(MM,NN)= A3
  STNL32(MM,NN) = B3
  STNL33(MM,NN) = C3
C
  NN=NN+1
30 CONTINUE
  MM=MM+1
  NN=1
10 CONTINUE
  RETURN
  END
C******************************************************************
C
  SUBROUTINE CONVRGE(NCV,ITER,IMAX,KTINC)
C
C******************************************************************
  IMPLICIT DOUBLE PRECISION (A-H,O-Z)
  COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
 X   STNL13(225,225),STNL23(225,225),STNL31(225,225),
 X   STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
  COMMON/ELAS/NANAL(3),E1,E2,G12,NU12,PT,THE(20),NP,A(3,3),
 X       B(3,3),D(3,3),E(3,3),F(3,3),H(3,3),
 X       AS(2,2),DS(2,2),FS(2,2),G13,
 X       G23,EY,NU,HT,RTHE(20),GS
  COMMON/CONVERGE/DIS(2205),DISPREV(2205)
  DOUBLE PRECISION K1,NU,NU12,NU21
  IF(NANAL(2).EQ.1) NCV=1
  IF(NANAL(2).EQ.1)RETURN
  IF(ITER.EQ.0)THEN
  ITER=1
  NCV=0
  RETURN
  ELSE
  PRINT*,' KTINC= ',KTINC
  PRINT*,' ITER= ',ITER
  ENDIF
  PNORM1=0.
  PNORM2=0.
  PNORM3=0.
  PNORM4=0.
  NT=5*NTERMS*NTERMS
C ADD DELTA D TO D
  DO 5 I=1,NT
5 DIS(I)=DISPREV(I)+DIS(I)
  DO 10 I=2*NTERMS*NTERMS,NT
  PNORM1=DABS(DIS(I))**2+PNORM1
10 PNORM2=DABS(DIS(I)-DISPREV(I))**2+PNORM2
  TOL=PNORM2/PNORM1*100
  DO 30 I=1,2*NTERMS*NTERMS
  PNORM3=DABS(DIS(I))**2 + PNORM3
```

```
 30 PNORM4=DABS(DIS(I)-DISPREV(I))**2+PNORM4
   TOL1=PNORM4/PNORM3*100.
   PRINT*,'TOL= ',TOL
   PRINT*,'TOL1= ',TOL1
   IF (TOL .GT. .05 .OR. TOL1 .GT. 1. .OR. ITER .LE. 2) THEN
   ITER=ITER+1
    IF(ITER.GT.IMAX)THEN
     PRINT*,' MAX ITERATIONS'
     STOP
    ELSE
    RETURN
    ENDIF
   ELSE
   NCV=1
   WRITE(6,849)
 849 FORMAT(//)
   WRITE(6,850)KTINC,ITER
 850 FORMAT(1X,'INCREMENT= ',I3,' ITERATION= ',I2)
   WRITE(6,849)
C
C BUMP INCREMENT COUNT, RESET ITER
C NOTE: NCV IS RESET TO ZERO W/N DISPLACEMENT
C
   KTINC=KTINC+1
   ITER=1
C INCREMENT LOAD
   DO 20 I=1,NT
 20 PLOAD(I)=PLOAD(I)*KTINC/(KTINC-1)
   ENDIF
   RETURN
   END
C*******************************************
C
   SUBROUTINE DISPLACEMENT(NTERMS,NCV,HT,PHO,R,S,NANAL)
C
C*******************************************
   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
   COMMON/CONVERGE/DIS(2205),DISPREV(2205)
   COMMON/STRS/Q11TOP,Q12TOP,Q22TOP,Q16TOP,Q26TOP,
  X     QS11,QS22,Q12,Q22
   DOUBLE PRECISION K1,NU,NU12,NU21
   PI=3.14159265359
   NT=NTERMS*NTERMS
   K1=-4./(3.*HT**2)
   U=0.
   V=0.
   W=0.
   KT=1
C $$$$$
   DO 10 I=1,NTERMS
   DO 20 J=1,NTERMS
   W=DSIN(I*PI/2)*DSIN(J*PI/2)*DIS(KT+2*NT)+W
   U=DCOS(I*PI)*DSIN(J*PI/2)*DIS(KT)+U
   V=DSIN(I*PI/2)*DCOS(J*PI)*DIS(KT+NT)+V
   KT=KT+1
 20 CONTINUE
 10 CONTINUE
   PRINT*,' NCV= ',NCV,' W CENTER = ',W
   IF(NCV.EQ.0)RETURN
   WRITE(6,900)W
 900 FORMAT(1X,'W CENTER = ',D20.13)
   WRITE(6,901)U
 901 FORMAT(1X,'U(R,S/2)= ',D20.13)
   WRITE(6,902)V
 902 FORMAT(1X,'V(R/2,S)= ',D20.13)
C CALCULATE STRESS
   EXT=0.
   EYT=0.
   EXYT=0.
   EXYB=0.
   EXB=0.
```

```
      EYB=0.
      ES4=0.
      ES5=0.
      EXX=0.
      EYY=0.
      T2=HT/2
      T4=HT/4
      KT3=1
C $$$$$
      DO 30 I=1,NTERMS
      DO 40 J=1,NTERMS
      SI=DSIN(I*PI/2)*DSIN(J*PI/2)
      CO=DCOS(I*PI/2)*DCOS(J*PI/2)
C STRAIN X,Y,XY @ TOP/BOTTOM CENTER PLATE
      EXT=-I*PI/R*(DIS(KT3)-T2*DIS(KT3+3*NT)-T2**3*K1*
     X(DIS(KT3+3*NT)+I*PI/R*DIS(KT3+2*NT)))*SI+EXT
      EYT=(-J*PI/S*(DIS(KT3+NT)-T2*DIS(KT3+4*NT)-T2**3*K1*
     X(DIS(KT3+4*NT)+J*PI/S*DIS(KT3+2*NT)))-DIS(KT3+2*NT)*PHO)*SI+EYT
C**********************************************************************
      EXYT=(J/S*DIS(KT3)+I/R*DIS(KT3+NT)-T2*(J/S*DIS(KT3+3*NT)
     X+I/R*DIS(KT3+4*NT))-T2**3*K1*(J/S*DIS(KT3+3*NT)+I/R*DIS(KT3+4*NT)
     X+2*I/R*J/S*DIS(KT3+2*NT)))*PI*CO+EXYT
      EXYB=(J/S*DIS(KT3)+I/R*DIS(KT3+NT)+T2*(J/S*DIS(KT3+3*NT)
     X+I/R*DIS(KT3+4*NT))+T2**3*K1*(J/S*DIS(KT3+3*NT)+I/R*DIS(KT3+4*NT)
     X+2*I/R*J/S*DIS(KT3+2*NT)))*PI*CO+EXYB
C**********************************************************************
      EXB=-I*PI/R*(DIS(KT3)+T2*DIS(KT3+3*NT)+T2**3*K1*
     X(DIS(KT3+3*NT)+I*PI/R*DIS(KT3+2*NT)))*SI+EXB
      EYB=(-J*PI/S*(DIS(KT3+NT)+T2*DIS(KT3+4*NT)+T2**3*K1*
     X(DIS(KT3+4*NT)+J*PI/S*DIS(KT3+2*NT)))-DIS(KT3+2*NT)*PHO)*SI+EYB
C SHEAR STRAINS @ Z=0.
      ES4=(DIS(KT3+4*NT)+J*PI/S*DIS(KT3+2*NT))*
     XDSIN(I*PI/2)*DCOS(J*PI)+ES4
      ES5=(DIS(KT3+3*NT)+I*PI/R*DIS(KT3+2*NT))*
     XDCOS(I*PI)*DSIN(J*PI/2)+ES5
C STRAIN X,Y @ Z=T/4
      EXX=-I*PI/R*(DIS(KT3)+T4*DIS(KT3+3*NT)+T4**3*K1*
     X(DIS(KT3+3*NT)+I*PI/R*DIS(KT3+2*NT)))*SI+EXX
      EYY=(-J*PI/S*(DIS(KT3+NT)+T4*DIS(KT3+4*NT)+T4**3*K1*
     X(DIS(KT3+4*NT)+J*PI/S*DIS(KT3+2*NT)))-DIS(KT3+2*NT)*PHO)*SI+EYY
C
      KT3=KT3+1
   40 CONTINUE
   30 CONTINUE
C SXT AT (R/2,S/2,-H/2), SY AT (R/2,S/2,H/4), SXY AT (0,0,H/2)
C SS4 AT (R/2,S,0), SS5 AT (R,S/2,0)
      SXT=Q11TOP*EXT + Q12TOP*EYT + Q16TOP*EXYT
      SXB=Q11TOP*EXB + Q12TOP*EYB + Q16TOP*EXYB
      SYT=Q12TOP*EXT + Q22TOP*EYT + Q26TOP*EXYT
      SYB=Q12TOP*EXB + Q22TOP*EYB + Q26TOP*EXYB
      SS4=QS11*ES4
      SS5=QS22*ES5
      SY=Q12*EXX+Q22*EYY
      WRITE(6,905) SXT
  905 FORMAT(1X,'SIGMA-X AT -Z = ',D20.13)
      WRITE(6,906) SXB
  906 FORMAT(1X,'SIGMA-X AT +Z = ',D20.13)
      WRITE(6,907) SYT
  907 FORMAT(1X,'SIGMA-Y AT -Z = ',D20.13)
      WRITE(6,908) SYB
  908 FORMAT(1X,'SIGMA-Y AT +Z = ',D20.13)
      WRITE(6,911) SY
  911 FORMAT(1X,'SIGMA-Y AT H/4 = ',D20.13)
      WRITE(6,909) SS4
  909 FORMAT(1X,'SIGMA-4 AT (R/2,S,0) = ',D20.13)
      WRITE(6,910) SS5
  910 FORMAT(1X,'SIGMA-5 AT (R,S/2,0) = ',D20.13)
C
C RESET CONVERGENCE FLAG
C
      NCV=0
```

```
      RETURN
      END
C 4NOV94
C**********************************************************************
C
      SUBROUTINE INPUT(NPRNT,NINC,IMAX)
C
C**********************************************************************
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
     X    STNL13(225,225),STNL23(225,225),STNL31(225,225),
     X    STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
      COMMON/ELAS/NANAL(3),E1,E2,G12,NU12,PT,THE(20),NP,A(3,3),
     X          B(3,3),D(3,3),E(3,3),F(3,3),H(3,3),
     X          AS(2,2),DS(2,2),FS(2,2),G13,
     X          G23,EY,NU,HT,RTHE(20),GS
      DOUBLE PRECISION K1,NU,NU12,NU21
C     READ(5,910) TITLE
C 910 FORMAT (20A4)
      READ(5,*) NANAL(1),NANAL(2)
      IF(NANAL(2).EQ.2)READ(5,*) NINC,IMAX
      READ(5,*)NTERMS
      READ(5,*) NPRNT
      READ(5,*) NLOAD,Q0
C
C     ECHO INPUT
C
      WRITE(6,896)
 896  FORMAT(//)
      WRITE(6,897)
 897  FORMAT(1X,'NANAL(1)=0,1,2 FOR ARB,ISO,SYM')
      WRITE(6,895)
 895  FORMAT(1X,'NANAL(2)=0,1,2 FOR NL,LIN,EIGEN')
      WRITE(6,899)NANAL(1),NANAL(2)
 899  FORMAT(1X,'NANAL(1)= ',I1,'  NANAL(2)= ',I2)
      IF(NANAL(2).EQ.2)WRITE(6,894)NINC
 894  FORMAT(1X,' NUMBER OF LOAD INCREMENTS= ',I3)
      WRITE(6,896)
      WRITE(6,898)NTERMS
 898  FORMAT(1X,'NUMBER OF TERMS IN SERIES (ODD/EVEN)= ',I2)
      WRITE(6,896)
      WRITE(6,893)NLOAD
 893  FORMAT(1X,'NLOAD(0=SINUSOIDAL, 1=UNIFORM, 2=CTR PT LOAD) = ',I2)
      WRITE(6,892)Q0
 892  FORMAT(1X,'MAGNITUDE OF LOAD = ',D20.13)
      WRITE(6,896)
      IF (NANAL(1).EQ.2) GOTO 75
C
C FOR ISOTROPIC
C
      READ(5,*)EY,NU,HT
      WRITE(6,901)
 901  FORMAT(1X,'THE FOLLOWING PROPERTIES WERE INPUT (E,NU,THICK)')
      WRITE(6,906)EY
      WRITE(6,908)NU
      WRITE(6,906)HT
      GOTO 78
C
C FOR ORTHOTROPIC, INPUT MATERIAL PROPERTIES, E1,E2,G12,NU12
C
 75   READ(5,*) E1,E2,G12,NU12,G13,G23
      WRITE(6,904)
 904  FORMAT(1X,'THE FOLLOWING PROPERTIES WERE INPUT')
      WRITE(6,905)
 905  FORMAT(1X,'E1,E2,G12,NU12,G13,G23')
      WRITE(6,906)E1
      WRITE(6,906)E2
      WRITE(6,906)G12
      WRITE(6,908)NU12
      WRITE(6,906)G13
      WRITE(6,906)G23
```

```
906 FORMAT(1X,D20.13)
908 FORMAT(1X,D20.13)
  READ(5,*)NP,PT
  READ(5,*) (THE(II),II=1,NP)
  WRITE(6,916)
916 FORMAT(1X)
  WRITE(6,918)
918 FORMAT(1X,'THE FOLLOWING LAMINATE WAS INPUT')
  DO 76 II=1,NP
 76 WRITE(6,920)THE(II)
920 FORMAT(1X,D20.13)
  WRITE(6,916)
  WRITE(6,922) PT
922 FORMAT(1X,'PLY THICKNESS = ',D20.13)
 78 READ(5,*)R,S,RAD
  WRITE(6,919)R,S,RAD
  IF(RAD.EQ.0.)THEN
   PHO=0.
  ELSE
   PHO=1./RAD
  ENDIF
919 FORMAT(1X,'PANEL',D20.13,' X ',D20.13,' RAD= ',D20.13)
  WRITE(6,896)
  RETURN
  END
C***********************************************************************
C
  SUBROUTINE ELAST(NPRNT,S,PHO)
C
C***********************************************************************
C....................................................................
C  THIS SUBROUTINE CALCULATES THE ELASTICITY MATRICES, A,B,D,E,F
C  H,AS,DS,FS
C....................................................................
  IMPLICIT DOUBLE PRECISION (A-H,O-Z)
  COMMON/ELAS/NANAL(3),E1,E2,G12,NU12,PT,THE(20),NP,A(3,3),
 X      B(3,3),D(3,3),E(3,3),F(3,3),H(3,3),
 X      AS(2,2),DS(2,2),FS(2,2),G13,
 X      G23,EY,NU,HT,RTHE(20),GS
  COMMON/STRS/Q11TOP,Q12TOP,Q22TOP,Q16TOP,Q26TOP,
 X      QS11,QS22,Q12,Q22
  DOUBLE PRECISION K1,NU,NU12,NU21
  DIMENSION QBAR(3,3),QSBAR(2,2)
C
C INITIALIZE THE ELASTICITY MATRICES
C
  DO 10 M=1,3
  DO 11 N=1,3
  A(M,N)=0.
  B(M,N)=0.
  D(M,N)=0.
  E(M,N)=0.
  F(M,N)=0.
 11 H(M,N)=0.
 10 CONTINUE
  DO 15 M=1,2
  DO 16 N=1,2
  AS(M,N)=0.
  DS(M,N)=0.
 16 FS(M,N)=0.
 15 CONTINUE
  IF(NANAL(1).NE.1)GOTO 30
C
C ISOTROPIC CASE
C
  GS=EY/(2.*(1.+NU))
  DENOM=1.-NU**2
  QBAR(1,1)=EY/DENOM
  QBAR(1,2)=NU*EY/DENOM
  QBAR(2,2)=QBAR(1,1)
  QBAR(2,1)=QBAR(1,2)
```

```
      QBAR(3,3)=GS
      QBAR(1,3)=0.
      QBAR(3,1)=0.
      QBAR(2,3)=0.
      QBAR(3,2)=0.
      QSBAR(1,1)=GS
      QSBAR(2,2)=GS
      QSBAR(1,2)=0.
      QSBAR(2,1)=0.
      Q11TOP=QBAR(1,1)
      Q12TOP=QBAR(1,2)
      Q22TOP=QBAR(2,2)
      QS11=QSBAR(1,1)
      QS22=QSBAR(2,2)
      QS12=0.
      Q22=QBAR(2,2)
      DO 20 M=1,3
      DO 21 N=1,3
      A(M,N)=QBAR(M,N)*HT
      D(M,N)=QBAR(M,N)*HT**3/(3*2.**2)
      F(M,N)=QBAR(M,N)*HT**5/(5*2.**4)
 21   H(M,N)=QBAR(M,N)*HT**7/(7*2.**6)
 20   CONTINUE
      DO 25 M=1,2
      DO 26 N=1,2
      AS(M,N)=QSBAR(M,N)*HT
      DS(M,N)=QSBAR(M,N)*HT**3/(3*2.**2)
 26   FS(M,N)=QSBAR(M,N)*HT**5/(5*2.**4)
 25   CONTINUE
      GOTO 29
C
C CALCULATE REDUCED STIFFNESSES
C FOR LAMINATED ANISOTROPIC STRUCTURES
C
 30   HT=PT*NP
      NU21=E2*NU12/E1
      DENOM=1.-NU12*NU21
      Q11=E1/DENOM
      Q12=NU12*E2/DENOM
      Q22=E2/DENOM
C
C CALCULATE INVARIANTS
C
      U1=(3.*Q11+3.*Q22+2.*Q12+4.*G12)/8.
      U2=(Q11-Q22)/2.
      U3=(Q11+Q22-2.*Q12-4.*G12)/8.
      U4=(Q11+Q22+6.*Q12-4.*G12)/8.
      U5=(Q11+Q22-2.*Q12+4.*G12)/8.
C****************************************************
C CALCULATE THE ELASTICITY MATRICES              *
C                                                *
C REMEM THAT THE Z AXIS POINTS DOWN AS IN JONES  *
C HOWEVER, THE FIRST PLY IS THE TOP PLY, IE,     *
C THE PLY WITH THE MOST NEGATIVE Z !!!           *
C****************************************************
      DO 45 II=1,NP
 45   RTHE(II)=THE(II)*3.14159265359/180.
      DO 50 KK=1,NP
      QBAR(1,1)=U1 + U2*DCOS(2.*RTHE(KK)) + U3*DCOS(4.*RTHE(KK))
      QBAR(1,2)=U4 - U3*DCOS(4.*RTHE(KK))
      QBAR(2,2)=U1 - U2*DCOS(2.*RTHE(KK)) + U3*DCOS(4.*RTHE(KK))
      QBAR(1,3)=.5*U2*DSIN(2.*RTHE(KK)) + U3*DSIN(4.*RTHE(KK))
      QBAR(2,3)=.5*U2*DSIN(2.*RTHE(KK)) - U3*DSIN(4.*RTHE(KK))
      QBAR(3,3)=U5 - U3*DCOS(4.*RTHE(KK))
      QBAR(2,1)=QBAR(1,2)
      QBAR(3,1)=QBAR(1,3)
      QBAR(3,2)=QBAR(2,3)
      QSBAR(1,1)=G23*DCOS(RTHE(KK))**2.+G13*DSIN(RTHE(KK))**2.
      QSBAR(2,2)=G13*DCOS(RTHE(KK))**2.+G23*DSIN(RTHE(KK))**2.
      QSBAR(1,2)=-(G23-G13)*DCOS(RTHE(KK))*DSIN(RTHE(KK))
      QSBAR(2,1)=QSBAR(1,2)
```

```
      IF(KK.EQ.1)THEN
       Q11TOP=QBAR(1,1)
       Q12TOP=QBAR(1,2)
       Q22TOP=QBAR(2,2)
       Q16TOP=QBAR(1,3)
       Q26TOP=QBAR(2,3)
       ELSE
       ENDIF
      IF(KK.EQ.NP/2+1)THEN
      QS11=QSBAR(1,1)
      QS22=QSBAR(2,2)
      ELSE
      ENDIF
      IF(KK.EQ.3)THEN
      Q12=QBAR(1,2)
      Q22=QBAR(2,2)
      ELSE
      ENDIF
      ZL=(KK*1. - NP*.5)*PT
      ZU=ZL-PT
   57 DO 51 M=1,3
      DO 52 N=1,3
      A(M,N)=A(M,N) + QBAR(M,N)*PT
      D(M,N)=D(M,N) + QBAR(M,N)*(ZL**3-ZU**3)/3.
      F(M,N)=F(M,N) + QBAR(M,N)*(ZL**5-ZU**5)/5.
      H(M,N)=H(M,N) + QBAR(M,N)*(ZL**7-ZU**7)/7.
      IF (NANAL(1).EQ.2) GOTO 52
      B(M,N)=B(M,N) + QBAR(M,N)*(ZL**2-ZU**2)/2.
      E(M,N)=E(M,N) + QBAR(M,N)*(ZL**4-ZU**4)/4.
   52 CONTINUE
   51 CONTINUE
      DO 60 M=1,2
      DO 61 N=1,2
      AS(M,N)=AS(M,N)+QSBAR(M,N)*PT
      DS(M,N)=DS(M,N)+QSBAR(M,N)*(ZL**3-ZU**3)/3.
   61 FS(M,N)=FS(M,N)+QSBAR(M,N)*(ZL**5-ZU**5)/5.
   60 CONTINUE
   50 CONTINUE
C
C SET TO ZERO THOSE ENTRIES DUE TO ROUNDOFF ERROR
C
   29 DO 85 M=1,3
      DO 86 N=1,3
      IF(DABS(A(1,1)).GT.DABS(A(M,N)*1.D08))A(M,N)=0.
      IF(DABS(B(1,1)).GT.DABS(B(M,N)*1.D08))B(M,N)=0.
      IF(DABS(D(1,1)).GT.DABS(D(M,N)*1.D08))D(M,N)=0.
      IF(DABS(E(1,1)).GT.DABS(E(M,N)*1.D08))E(M,N)=0.
      IF(DABS(F(1,1)).GT.DABS(F(M,N)*1.D08))F(M,N)=0.
      IF(DABS(H(1,1)).GT.DABS(H(M,N)*1.D08))H(M,N)=0.
   86 CONTINUE
   85 CONTINUE
      DO 90 M=1,2
      DO 91 N=1,2
      IF(DABS(AS(1,1)).GT.DABS(AS(M,N)*1.D08))AS(M,N)=0.
      IF(DABS(DS(1,1)).GT.DABS(DS(M,N)*1.D08))DS(M,N)=0.
      IF(DABS(FS(1,1)).GT.DABS(FS(M,N)*1.D08))FS(M,N)=0.
   91 CONTINUE
   90 CONTINUE
C CALCULATE BATDORF-STEIN PARAMETER
      BAT1=SQRT(A(1,1)*A(2,2)-A(1,2)**2)
      BAT2=SQRT(12.*SQRT(A(1,1)*A(2,2)*D(1,1)*D(2,2)))
      BATDORF=S**2*BAT1/BAT2*PHO
      WRITE(6,962)BATDORF
  962 FORMAT(1X,'BATDORF PARAMETER = ',D20.13)
      IF(NPRNT.EQ.0)RETURN
C
C OUTPUT THE MATRICES
C
      WRITE(6,916)
  916 FORMAT (1X)
      WRITE(6,950)
```

```
 950 FORMAT(1X,'A(I,J)=')
   DO 65 II=1,3
 65  WRITE(6,952)A(II,1),A(II,2),A(II,3)
   WRITE(6,916)
   WRITE(6,954)
 954 FORMAT(1X,'B(I,J)=')
   DO 66 II=1,3
 66  WRITE(6,952)B(II,1),B(II,2),B(II,3)
   WRITE(6,916)
   WRITE(6,956)
 956 FORMAT(1X,'D(I,J)=')
   DO 67 II=1,3
 67  WRITE(6,952)D(II,1),D(II,2),D(II,3)
   WRITE(6,916)
   WRITE(6,958)
 958 FORMAT(1X,'E(I,J)=')
   DO 68 II=1,3
 68  WRITE(6,952)E(II,1),E(II,2),E(II,3)
   WRITE(6,916)
   WRITE(6,960)
 960 FORMAT(1X,'F(I,J)=')
   DO 69 II=1,3
 69  WRITE(6,952)F(II,1),F(II,2),F(II,3)
   WRITE(6,916)
   WRITE(6,964)
 964  FORMAT(1X,'H(I,J)= ')
   DO 71 II=1,3
 71  WRITE(6,952)H(II,1),H(II,2),H(II,3)
   WRITE(6,916)
   WRITE(6,982)
 982 FORMAT(1X,'AS(I,J)= ')
   DO 80 II=1,2
 80  WRITE(6,953)AS(II,1),AS(II,2)
   WRITE(6,916)
   WRITE(6,984)
 984  FORMAT(1X,'DS(I,J)= ')
   DO 81 II=1,2
 81  WRITE(6,953)DS(II,1),DS(II,2)
   WRITE(6,916)
   WRITE(6,986)
 986  FORMAT(1X,'FS(I,J)= ')
   DO 82 II=1,2
 82  WRITE(6,953)FS(II,1),FS(II,2)
 952  FORMAT(1X,3(D20.13,2X))
 953  FORMAT(1X,2(D20.13,2X))
   RETURN
   END
C*********************************************************
C
C SUBROUTINES FOR SOLVING SIMEQ, SOLVES STIFF*d=DIS
C
C*********************************************************
   SUBROUTINE LUDCMP
   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
   COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
  X     STNL13(225,225),STNL23(225,225),STNL31(225,225),
  X     STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
   COMMON/SOLVE/INDX(2205),VV(2205)
C
   N=5*NTERMS*NTERMS
   TINY=1.0D-20
   DD=1.
   DO 12 I=1,N
   AAMAX=0.
   DO 11 J=1,N
   IF(DABS(STIFF(I,J)).GT.AAMAX) AAMAX=DABS(STIFF(I,J))
 11 CONTINUE
   IF(AAMAX.EQ.0.) PAUSE 'SINGULAR MATRIX'
   VV(I)=1./AAMAX
 12 CONTINUE
C
```

```
      DO 19 J=1,N
      DO 14 I=1,J-1
      SUM=STIFF(I,J)
      DO 13 K=1,I-1
      SUM=SUM-STIFF(I,K)*STIFF(K,J)
   13 CONTINUE
      STIFF(I,J)=SUM
   14 CONTINUE
C
      AAMAX=0.
      DO 16 I=J,N
      SUM=STIFF(I,J)
      DO 15 K=1,J-1
      SUM=SUM-STIFF(I,K)*STIFF(K,J)
   15 CONTINUE
      STIFF(I,J)=SUM
      DUM=VV(I)*DABS(SUM)
      IF(DUM.GE.AAMAX) THEN
      IMAX=I
      AAMAX=DUM
      ENDIF
   16 CONTINUE
C
      IF (J.NE.IMAX)THEN
      DO 17 K=1,N
      DUM=STIFF(IMAX,K)
      STIFF(IMAX,K)=STIFF(J,K)
      STIFF(J,K)=DUM
   17 CONTINUE
      DD=-DD
      VV(IMAX)=VV(J)
      ENDIF
      INDX(J)=IMAX
      IF(STIFF(J,J).EQ.0.) STIFF(J,J)=TINY
      IF(J.NE.N)THEN
      DUM=1./STIFF(J,J)
      DO 18 I=J+1,N
      STIFF(I,J)=STIFF(I,J)*DUM
   18 CONTINUE
      ENDIF
   19 CONTINUE
C
      RETURN
      END
C
C
C
      SUBROUTINE LUBKSB
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
     X     STNL13(225,225),STNL23(225,225),STNL31(225,225),
     X     STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
      COMMON/SOLVE/INDX(2205),VV(2205)
      COMMON/CONVERGE/DIS(2205),DISPREV(2205)
      N=5*NTERMS*NTERMS
C
      II=0
      DO 12 I=1,N
      LL=INDX(I)
      SUM=DIS(LL)
      DIS(LL)=DIS(I)
      IF(II.NE.0)THEN
      DO 11 J=II,I-1
      SUM=SUM-STIFF(I,J)*DIS(J)
   11 CONTINUE
      ELSE IF (SUM.NE.0.) THEN
      II=I
      ENDIF
      DIS(I)=SUM
   12 CONTINUE
      DO 14 I=N,1,-1
```

```
      SUM=DIS(I)
      IF(I.LT.N)THEN
      DO 13 J=I+1,N
      SUM=SUM-STIFF(I,J)*DIS(J)
   13 CONTINUE
      ENDIF
      DIS(I)=SUM/STIFF(I,I)
   14 CONTINUE
      RETURN
      END
C*****************************************************************
C
      SUBROUTINE SUBST(NANAL,ITER)
C
C*****************************************************************
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/STF/R,S,RAD,ST(2205,2205),NTERMS,PLOAD(2205),Q0,NLOAD,PHO,
     X     STNL13(225,225),STNL23(225,225),STNL31(225,225),
     X     STNL32(225,225),STNL33(225,225),STIFF(2205,2205)
      COMMON/CONVERGE/DIS(2205),DISPREV(2205)
      NT=NTERMS*NTERMS
      IF(NANAL.EQ.1 .OR. ITER.EQ.0)GOTO 35
      DO 5 I=1,5*NT
    5 DISPREV(I)=DIS(I)
   35 CONTINUE
      DO 10 I=1,5*NT
      DO 20 J=1,5*NT
   20 STIFF(I,J)=ST(I,J)
   10 DIS(I)=PLOAD(I)
      IF(NANAL.EQ.1 .OR. ITER.EQ.0) RETURN
C FORM UPDATED STIFFNESS
      DO 30 I=1,NT
      DO 40 J=1,NT
      STIFF(I,2*NT+J)=STIFF(I,2*NT+J) + STNL13(I,J)
      STIFF(NT+I,2*NT+J)=STIFF(NT+I,2*NT+J)+STNL23(I,J)
      STIFF(2*NT+I,J)=STIFF(2*NT+I,J) + STNL31(I,J)
      STIFF(2*NT+I,NT+J)=STIFF(2*NT+I,NT+J) + STNL32(I,J)
   40 STIFF(2*NT+I,2*NT+J)=STIFF(2*NT+I,2*NT+J) + STNL33(I,J)
   30 CONTINUE
C SOLVE FOR RESIDUAL FORCES, PUT INTO ARRAY DIS
      DO 50 I=1,5*NT
      SUMMER=0.
      DO 60 J=1,5*NT
   60 SUMMER=-STIFF(I,J)*DISPREV(J)+SUMMER
   50 DIS(I)=SUMMER+PLOAD(I)
      RETURN
      END
```

SAMPLE INPUT/OUTPUT LISTINGS
SAMPLE MATCHES RESULTS OF TABLE 5, *M,N* = 13.

INPUT:

```
2,1
7
0
1,10000.
25.e6,1.e6,.5e6,.25,.5e6,.2e6
4,.025
0.,90.,90.,0.
10.,10.,100.
```

OUTPUT:

```
NANAL(1)=0,1,2 FOR ARB,ISO,SYM
NANAL(2)=0,1,2 FOR NL,LIN,EIGEN
NANAL(1)= 2  NANAL(2)= 1


NUMBER OF TERMS IN SERIES (ODD ONLY)= 7


NLOAD(0=SINUSOIDAL, 1=UNIFORM, 2=CTR PT LOAD) = 1
MAGNITUDE OF LOAD = 0.1000000000000D+05


THE FOLLOWING PROPERTIES WERE INPUT
E1,E2,G12,NU12,G13,G23
 0.2500000000000D+08
 0.1000000000000D+07
 0.5000000000000D+06
 0.2500000000000D+00
 0.5000000000000D+06
 0.2000000000000D+06

THE FOLLOWING LAMINATE WAS INPUT
 0.0000000000000D+00
 0.9000000000000D+02
 0.9000000000000D+02
 0.0000000000000D+00

PLY THICKNESS = 0.2500000000000D-01
PANEL 0.1000000000000D+02 X 0.1000000000000D+02  RAD= 0.1000000000000D+03


BATDORF PARAMETER = 0.1176984347640D+02

 NCV=  1 W CENTER =    566.09969091488

W CENTER = 0.5660996909149D+03
U(R,S/2)= -0.6378059133087D+00
V(R/2,S)= 0.1809982682698D+02
SIGMA-X AT -Z = -0.7283224645778D+08
SIGMA-X AT +Z = 0.6273132129261D+08
SIGMA-Y AT -Z = -0.3159826271727D+07
SIGMA-Y AT +Z = 0.2666074544675D+07
SIGMA-Y AT -H/4 = -0.3331700537210D+08
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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
April 1995

3. REPORT TYPE AND DATES COVERED
Final

4. TITLE AND SUBTITLE
A Galerkin Solution to Geometrically Nonlinear Shallow Shell Equations

5. FUNDING NUMBERS

6. AUTHOR(S)
Major Scott T. Dennis

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Department of Engineering Mechanics
USAFA/DFEM
USAF Academy CO 80840

8. PERFORMING ORGANIZATION REPORT NUMBER
USAFA-TR-95-3

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribtution unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

A laminated shallow shell approach that includes von Karman geometric nonlinearity and parabolic transverse shear deformation is posed in differential operator form. Trigonmetric series are assumed for each of the five shell displacement degrees freedom for the subsequent nonlinear galerkin solution resulting in $5n^2$ simultaneous algebraic equations where n is the number of displacement terms assumed in the series. The galerkin nonlinear solution is computationally intensive. The response of several laminate geometries subjected to transverse loadings are found. Thicker plates and shells generally exhibit more flexible response compared to thinner geometries in both linear and nonlinear analyses. The nondimensional shell response is examined by using the Batdorf-Stein shell parameter for laminated constructions. Quasi-isotropic shallow shells undergo significant transverse shear flexibility in the thicker geometries as given by the nondimensional shell crown deflection. However, the nondimensional crown deflection in the deeper shell response is not much influenced by shell thickness. For flat plates, geometric nonlinearity lessens the influence of transverse shear flexibility when compared to linear solutions due to membrane stretching resistance.

14. SUBJECT TERMS
Shell von Karman Transverse Shear
Laminate Donnell Nonlinear
Galerkin

15. NUMBER OF PAGES
64

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified			